

Available online at [www.sciencedirect.com](http://www.sciencedirect.com) **ScienceDirect**

J. Math. Anal. Appl. 326 (2007) 533–555

---

*Journal of*  
**MATHEMATICAL  
ANALYSIS AND  
APPLICATIONS**

---

[www.elsevier.com/locate/jmaa](http://www.elsevier.com/locate/jmaa)

# The torus related Riemann problem

Alip Mohammed <sup>a, b</sup><sup>a</sup> *I. Math. Institute, Free University Berlin, Arnimallee 3, 14195 Berlin, Germany*<sup>b</sup> *Department of Mathematics, Xinjiang University, Shenglilu 14, 830046 Ürümqi, PR China*

Received 2 June 2005

Available online 17 April 2006

Submitted by G. Chen

---

## Abstract

The Riemann jump problem is solved for analytic functions of several complex variables with the unit torus as the jump manifold. A well-posed formulation is given which does not demand any solvability conditions. The higher dimensional Plemelj–Sokhotzki formula for analytic functions in torus domains is established. The canonical functions of the Riemann problem for torus domains are represented and applied in order to construct solutions for both of the homogeneous and inhomogeneous problems. Thus contrary to earlier research the results are similar to the respective ones for just one variable. A connection between the Riemann and the Riemann–Hilbert boundary value problem for the unit polydisc is explained.

© 2006 Elsevier Inc. All rights reserved.

**Keywords:** Several complex variables; Holomorphic functions; Riemann jump problem; Riemann–Hilbert problems; Plemelj–Sokhotzki formula; Torus domains

---

## 1. Introduction

The motivation behind the study of the Riemann and Riemann–Hilbert problems in higher dimensional torus domains comes from both the theoretical significance and the numerous applications of their one-dimensional analogue from crack problems in engineering to analysis of Markov processes with a two-dimensional state space in queuing system theory.

The Riemann problem is interesting not only for theoretical reasons but also with respect to applications. On this topic numerous research has been done and rich results are achieved in the plane case [4,9,17,22]. They lead to the development of new promising techniques for the analysis of a large class of problems [6,16,18]. The Riemann problem is also tightly connected with

---

*E-mail address:* [mohammed@math.fu-berlin.de](mailto:mohammed@math.fu-berlin.de).

the Riemann–Hilbert problem which has been the solution provider for a vast array of problems in mathematics, mathematical physics and applied mathematics [7]. However, the studies of the Riemann problem are mainly restricted in the plane case and thus higher dimensional considerations are necessary.

It is well known that the polydisc and the ball in the higher dimensional space are typical different natural extensions of the disc in the complex plane. Problems of the ball are well studied, but problems of polydomains are almost untouched due to geometrical complexity and some special properties of the polydomain in the higher dimensional space [3]. In this paper we study the Riemann problem for polydomains.

In the higher dimensional space, in general, the zero sets of analytic functions of several complex variables can be connected and thus the index method which was vital in the one variable case is questionable to apply. There is also no any convincing higher dimensional analogue of the Plemelj–Sokhotzki formula for analytic functions in torus domains which is fundamental for finding solutions of the one-dimensional problem. To develop the higher dimensional Plemelj–Sokhotzki formula for analytic functions in torus domains, the existing one-dimensional theorem is far from being satisfactory. More deeper inside knowledge is needed. Because of these difficulties, although there were some papers about the Shilov boundary related special inhomogeneous Riemann problem in  $\mathbb{C}^n$  ( $n > 1$ ) [5,11,15], no one has given a solution which is constructed by the canonical function for true higher dimensional torus domains so far, except [10,12] for bi-disc domains. However, the latter results have been found incorrect or inadequate [2].

Among the previous studies there was no one which could work for solving the corresponding homogeneous problem. The reason is that every attempt was based on the ready form of the one-dimensional Cauchy kernel, the one-dimensional Plemelj–Sokhotzki formula and the one-dimensional Noether condition. They were repeatedly applied for the problem variable by variable. These techniques work well for one inner and its outer domain in the plane. To apply these techniques for torus domains the problem was considered variable by variable so that one inner and its outer domain are always available. Thus because of the Noether condition for analytic functions of the outer domain in the plane, analytic functions of some torus domains get more strict restrictions [11] than necessary and adequate [1,20].

The problem in the one variable case essentially is about one pair of analytic functions, i.e., about one pair of domains. In the case of a torus there are pairs of domains which could be identified neither as definitive inner nor outer domains, every pair has nothing to do with the others [20,21]. In this sense previous studies have treated an analytic function of a torus domain also in other irrelevant torus domains. Additionally in order to obtain values of an analytic function of a torus domain, its values on the whole boundary of the domain were needed [5,15] which is contradictory to the statement that an analytic function in a polydomain can be fully determined by the boundary values of this function on the torus [13]. The viewpoint of the previous studies is always one of the  $n$  variables rather than one pair of the  $2^n$  torus domains in  $\mathbb{C}^n$ .

Only very recently some kind of special Riemann and Riemann–Hilbert problems for holomorphic functions have been treated from a new perspective. Necessary and sufficient conditions for the existence of finitely linearly independent solutions and finitely many solvability conditions were derived [8]. For the Riemann problem they are a special subject of our consideration.

An interesting topic which was not considered by the former studies is the connection between the Riemann problem and the Riemann–Hilbert problem in the several variables case, although in the one variable case it is [4]. For this reason applying Fourier series method and analyzing the structure of boundary values of analytic functions, some complementary concepts about the Noether condition [11], for higher dimensional torus domains have been clarified [1,20], which

forced previous discussions [2,12] to make some compromises on the properties of analytic functions in general and so lead to some artificial assumptions. The rearranged form of the boundary values of holomorphic functions in polydomains by the modified Cauchy kernel [1,20] and the well-posed formulation of the Riemann problem are the key factors for the establishment of the connection between the two problems for polydomains.

Taking all these above facts into account and taking advantage of the geometrical nature of the Shilov boundary we treat the problem applying the Fourier series method in the Hölder function space.

For certain reason we need to define a set of complex-valued functions

$$W(\mathbb{D}, \mathbb{C}) = \left\{ f \mid f(\zeta) = \sum_{k=-\infty}^{+\infty} a_k \zeta^k, \zeta \in \partial\mathbb{D}, \|f\|_W := \sum_{k=-\infty}^{+\infty} |a_k| < \infty \right\}$$

which is called the one-dimensional *Wiener algebra* [19] and simply denoted by  $W^1$ . By the Weierstrass theorem the Fourier series of functions from the Wiener algebra are also uniformly convergent. Because of the independence of the variables on the Shilov boundary  $\partial_0\mathbb{D}^n$  ( $n > 1$ ), we have the Wiener algebra for torus as

$$W^n = \left\{ f \mid f(z) = \sum_{\kappa=-\infty}^{+\infty} a_{\kappa} z^{\kappa}, \zeta \in \partial_0\mathbb{D}^n, \|f\|_{W^n} := \sum_{\kappa=-\infty}^{+\infty} |a_{\kappa}| < \infty \right\}.$$

For the sake of simplicity the Wiener algebra is applied as the function space in some cases as we want to see the essence of the one-dimensional problem so that we could reach a better understanding of the higher dimensional problem.

In [23] a special Riemann problem was studied in a higher dimensional space. Having difficulties with the resolution their result was restricted to the Wiener algebra. It was not possible to get the same result in  $C^{\alpha}$ ,  $0 < \alpha < 1$ . However according to the Bernstein theorem  $C^{\alpha}(\partial_0\mathbb{D}^n, \mathbb{C})$  turns out to be the Wiener algebra for  $\alpha > 1/2$  [14]. In this sense the Hölder function space with  $\alpha > 1/2$ , i.e., the Wiener algebra is applied only for clarifying the structure—the Cauchy kernel of the respective analytic functions in different torus domains [1,20]. Our discussion on the Riemann problem moreover will not be restricted to the Wiener algebra, but only to the Hölder function space  $C^{\alpha}(\partial_0\mathbb{D}^n, \mathbb{C})$  with  $0 < \alpha < 1$ . In this sense we have lost nothing at all compared with the one-dimensional case.

We begin our discussion with the original Riemann problem in the one variable case so that one can see the essence of the one-dimensional problem and the differences between the one and the higher dimensional cases easily.

Let  $D$  be a simply connected bounded domain with smooth boundary and  $G, g \in C^{\alpha}(\partial D, \mathbb{C})$ ,  $0 < \alpha < 1$ , with  $G(\zeta) \neq 0$  on  $\partial D$ . The Riemann problem demands to find analytic functions  $\phi^+$  in  $D^+ = D$  and  $\phi^-$  in  $D^- = \mathbb{C} \cup \{\infty\} \setminus (D^+ \cup \partial D)$  such that

$$\phi^+(\zeta) = G(\zeta)\phi^-(\zeta) + g(\zeta), \quad \zeta \in \partial D. \quad (1)$$

Here two functions on the boundary are given and two functions, analytic in two domains, have to be found. These two domains are uniquely determined by the given boundary and they cover the whole plane.

This problem is well studied and there are numerous results [4,9,17,22]. The following is the most important one.

**Theorem 1.** For  $0 \leq \kappa$  the homogeneous Riemann boundary value problem ( $g = 0$ ) has  $\kappa + 1$  linearly independent solutions:

$$\begin{aligned}
\phi_k^+(z) &= z^k e^{\gamma^+(z)}, \quad z \in D^+; \\
\phi_k^-(z) &= z^{k-\kappa} e^{\gamma^-(z)}, \quad z \in D^-, \quad 0 \leq k \leq \kappa; \\
\gamma(z) &:= \frac{1}{2\pi i} \int_{\partial D} \log\{\zeta^{-\kappa} G(\zeta)\} \frac{d\zeta}{\zeta - z}, \quad z \notin \partial D, \quad \kappa := \text{ind } G = \frac{1}{2\pi i} \int_{\partial D} d \log G(\zeta). \quad (2)
\end{aligned}$$

The general solution contains  $\kappa + 1$  arbitrary complex constants. For  $\kappa < 0$  the homogeneous problem ( $g = 0$ ) is only trivially solvable.

The function

$$X(z) = \begin{cases} X^+(z) = e^{\gamma^+(z)}, & z \in D^+, \\ X^-(z) = z^{-\kappa} e^{\gamma^-(z)}, & z \in D^-, \end{cases}$$

is called *canonical function of the RIEMANN problem*.

Clearly Eq. (2) can be written equally as

$$\begin{cases} \phi_\kappa^+(z) = P_\kappa(z) e^{\gamma^+(z)} = P_\kappa(z) X^+(z), & z \in D^+, \\ \phi_\kappa^-(z) = P_\kappa(z) z^{-\kappa} e^{\gamma^-(z)} = P_\kappa(z) X^-(z), & z \in D^-, \end{cases} \quad (3)$$

where  $P_\kappa(z)$  is a polynomial in  $D^+$  at most of degree  $\kappa$  with arbitrary coefficients. However  $P_\kappa(z)$  is not a polynomial in  $D^-$ . A polynomial in  $D^-$  at most of degree  $\kappa$  with arbitrary coefficients must look like  $P_\kappa(1/z)$  in order to behave regular at infinity. So it is more meaningful to write down Eq. (2) as

$$\begin{cases} \phi_\kappa^+(z) = P_\kappa(z) e^{\gamma^+(z)} =: P_\kappa^+(z) X_0^+(z), & z \in D^+, \\ \phi_\kappa^-(z) = P_\kappa(z^{-1}) e^{\gamma^-(z)} =: P_\kappa^-(z) X_0^-(z), & z \in D^-. \end{cases} \quad (4)$$

Of course Eqs. (3) is equivalent to Eqs. (4).

More detailed discussions about the Plemelj–Sokhotzki formula, the solutions of the inhomogeneous Riemann problem and the connection between the Riemann and the Riemann–Hilbert problem can be found again in [4,9,17,22].

In the following section we establish the Plemelj–Sokhotzki formula for torus domains.

## 2. The Plemelj–Sokhotzki formula and the well-posed formulation of the Riemann problem for torus domains

It is well known that the Plemelj–Sokhotzki formula is a key factor for achieving the canonical function of the Riemann problem and so for constructing the solutions both of the homogeneous and the inhomogeneous Riemann problem in plane domains. For the polydomains one has to find a proper method to obtain and prove the Plemelj–Sokhotzki formulas.

### 2.1. A simple proof of the Plemelj–Sokhotzki formula for the disc

We give first a simple proof of the formula for one variable.

Let  $\varphi \in W(\partial\mathbb{D}, \mathbb{C})$ , i.e.,

$$\varphi(\zeta) = \sum_{-\infty}^{+\infty} \alpha_k \zeta^k, \quad \zeta \in \partial\mathbb{D}, \quad \sum_{-\infty}^{+\infty} |\alpha_k| < +\infty. \quad (5)$$

We denote

$$\varphi(\zeta) = \sum_{-\infty}^{+\infty} \alpha_k \zeta^k = \sum_{k=0}^{\infty} \alpha_k \zeta^k + \sum_{k=1}^{\infty} \alpha_{-k} \zeta^{-k} =: \varphi^+(\zeta) + \varphi^-(\zeta), \quad \zeta \in \partial\mathbb{D}.$$

Then from (2) for  $z \in \mathbb{D}$  we have

$$\begin{aligned} \phi(z) &= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \varphi(\zeta) \frac{1}{1 - z\bar{\zeta}} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \left[ \sum_{-\infty}^{+\infty} \alpha_k \zeta^k \right] \left[ \sum_{h=0}^{+\infty} (z\bar{\zeta})^h \right] \frac{d\zeta}{\zeta} \\ &= \sum_{k=0}^{+\infty} \alpha_k z^k, \quad z \in \mathbb{D}, \end{aligned}$$

and for  $z \in \mathbb{D}^-$ ,

$$\begin{aligned} \phi(z) &= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \varphi(\zeta) \frac{z^{-1}}{z^{-1}\bar{\zeta} - 1} d\zeta = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \varphi(\zeta) \left[ -\frac{z^{-1}\bar{\zeta}}{1 - z^{-1}\bar{\zeta}} \right] \frac{d\zeta}{\zeta} \\ &= -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \left[ \sum_{-\infty}^{+\infty} \alpha_k \zeta^k \right] \left[ \sum_{h=1}^{+\infty} (z^{-1}\bar{\zeta})^h \right] \frac{d\zeta}{\zeta} = -\sum_{k=1}^{+\infty} \alpha_{-k} z^{-k}, \quad z \in \mathbb{D}^-. \end{aligned}$$

Again due to  $\varphi \in W(\partial\mathbb{D}, \mathbb{C})$  it is clear that for  $\eta \in \partial\mathbb{D}$ ,

(i) the series

$$\sum_{k=0}^{+\infty} \alpha_k \eta^k \quad \text{and} \quad \sum_{k=1}^{+\infty} \alpha_{-k} \eta^{-k}$$

are absolutely and uniformly convergent. Therefore

$$\phi^+(\eta) := \lim_{\substack{z \rightarrow \eta \\ z \in D^+}} \phi(z) = \sum_{k=0}^{+\infty} \alpha_k \eta^k \quad \text{and} \quad \phi^-(\eta) := \lim_{\substack{z \rightarrow \eta \\ z \in D^-}} \phi(z) = -\sum_{k=1}^{+\infty} \alpha_{-k} \eta^{-k},$$

further this means

$$\phi^+(\eta) - \phi^-(\eta) = \varphi(\eta), \quad \eta \in \partial\mathbb{D}; \quad (6)$$

(ii)  $\phi(\eta)$  exists for  $\eta \in \partial\mathbb{D}$  as Cauchy principal value and

$$\phi(\eta) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \varphi(\zeta) \frac{d\zeta}{\zeta - \eta} = \sum_{k=0}^{\infty} \alpha_k \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \zeta^k \frac{d\zeta}{\zeta - \eta} + \sum_{k=1}^{\infty} \alpha_{-k} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \zeta^{-k} \frac{d\zeta}{\zeta - \eta}.$$

However

$$\sum_{k=0}^{\infty} \alpha_k \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \zeta^k \frac{d\zeta}{\zeta - \eta} = \frac{1}{2} \sum_{k=0}^{\infty} \alpha_k \eta^k, \quad \eta \in \partial\mathbb{D},$$

and

$$\begin{aligned} \sum_{k=1}^{\infty} \alpha_{-k} \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \zeta^{-k} \frac{d\zeta}{\zeta - \eta} &= \sum_{k=1}^{\infty} \alpha_{-k} \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \left[ \left( \frac{1}{\zeta - \eta} - \frac{1}{\zeta} \right) \frac{1}{\eta^k} - \sum_{\ell=2}^k \bar{\zeta}^{\ell} \eta^{\ell-k-1} \right] d\zeta \\ &= \sum_{k=1}^{\infty} \alpha_{-k} \frac{1}{\eta^k} \left( \frac{1}{2} - 1 \right) = -\frac{1}{2} \sum_{k=1}^{\infty} \alpha_{-k} \eta^{-k}, \quad \eta \in \partial \mathbb{D}. \end{aligned}$$

Clearly

$$2\phi(\zeta) = \sum_{k=0}^{\infty} \alpha_k \zeta^k - \sum_{k=1}^{\infty} \alpha_{-k} \zeta^{-k} = \phi^+(\zeta) + \phi^-(\zeta), \quad \zeta \in \partial \mathbb{D}. \quad (7)$$

Further arbitrary given function  $\varphi \in W(\partial \mathbb{D}, \mathbb{C})$  is the sum of boundary values of two functions, analytic in  $\mathbb{D}^+$  and  $\mathbb{D}^-$ , respectively, i.e.,  $\varphi \in \partial \mathcal{H}^+ \oplus \partial \mathcal{H}^-$  where  $\partial \mathcal{H}^+$  ( $\partial \mathcal{H}^-$ ) is the set of the boundary values of functions which are analytic in  $\mathbb{D}^+$  ( $\mathbb{D}^-$ ). By (6) it is obvious that  $\phi^+(\zeta) \in \partial \mathcal{H}^+$  and  $\phi^-(\zeta) \in \partial \mathcal{H}^-$ .

The method applied here can be used to get a respective Plemelj–Sokhotzki formula for the torus related domains.

## 2.2. Formulation of the problem and the Plemelj–Sokhotzki formula

In order to get a better insight in the problem we begin our discussion at first with the two-dimensional case and the formulation of the problem.

The Riemann problem so far studied in the literature has the following formulation.

**Problem R<sub>1</sub>.** Let  $G^{++}, G^{+-}, G^{-+}, G^{--}, g \in W^2$ , with  $G^{++}(\zeta), G^{+-}(\zeta), G^{-+}(\zeta), G^{--}(\zeta) \neq 0$  on  $\partial_0 \mathbb{D}^2$ . Find analytic functions  $\phi^{++}, \phi^{+-}, \phi^{-+}, \phi^{--}$  in  $\mathbb{D}^{++}, \mathbb{D}^{+-}, \mathbb{D}^{-+}, \mathbb{D}^{--}$  such that

$$\begin{aligned} \phi^{++}(\zeta)G^{++}(\zeta) + \phi^{+-}(\zeta)G^{+-}(\zeta) + \phi^{-+}(\zeta)G^{-+}(\zeta) + \phi^{--}(\zeta)G^{--}(\zeta) &= g(\zeta), \\ \zeta &\in \partial_0 \mathbb{D}^2, \end{aligned} \quad (8)$$

where  $W^2$  is the Wiener algebra on  $\partial_0 \mathbb{D}^2$ .

Here we have four different domains divided by the torus and we have to find four analytic functions in the respective domains. However, to our understanding for a given value, an analytic function that can be defined in a respective torus domain has nothing in common with the analytic functions defined in the other respective torus domains, except with the analytic function defined in the totally opposed torus domain [1,20]. In this sense Eq. (8) has put two totally different kind of functions together taking into account that in the one variable case there is one equation on the boundary. In that case there is one boundary, one inner domain and one outer domain. One can say that there is one equation for one pair of domains. However, for  $\partial_0 \mathbb{D}^2$  there are two pairs of domains—not only one inner domain  $\mathbb{D}^{++}$  and its outer domain  $\mathbb{D}^{--}$  but there are also two other neighboring domains  $\mathbb{D}^{+-}, \mathbb{D}^{-+}$  [20,21] and if we take  $\mathbb{D}^{+-}$  as inner domain then  $\mathbb{D}^{-+}$  can be treated as its outer domain.

So the natural question is why there should be just only one equation for two pairs of domains? Of course (8) can be seen as one kind of higher dimensional analogue of the one-dimensional problem. As a consequence this version of the problem implies that only  $\mathbb{D}^{++}$  is the inner domain and all the others are treated as the outer domain of  $\mathbb{D}^{++}$ . This means however, the problem

should be treated then on  $\partial\mathbb{D}_1 \times \partial\mathbb{D}_2 \cup \partial\mathbb{D}_1 \times \mathbb{D}_2 \cup \mathbb{D}_1 \times \partial\mathbb{D}_2$ , i.e., on the whole boundary and not on the torus [5,15]. This then contradicts the conclusions of [13], i.e., any analytic function in polydiscs is determined only by its values on the torus but not by the values on the whole boundary. Although the other investigations have applied the Fourier method, they have used one-dimensional techniques and assumptions twice, i.e., the problem is treated once on  $\partial\mathbb{D}_1$  and once on  $\partial\mathbb{D}_2$ . That means again the problem is treated actually on the whole boundary.

Thus if we insist in one equation for one pair of domains which can be seen as the essence of the one-dimensional formulation of the problem, we can formulate the two-dimensional problem as follows.

**Problem G<sub>1</sub>.** Let  $G^{++}, G^{+-}, G^{-+}, G^{--}, g_1, g_2 \in W^2$  with  $G^{++}(\zeta), G^{+-}(\zeta), G^{-+}(\zeta), G^{--}(\zeta) \neq 0$  on  $\partial_0\mathbb{D}^2$ . Find analytic functions  $\phi^{++}, \phi^{+-}, \phi^{-+}, \phi^{--}$  in  $\mathbb{D}^{++}, \mathbb{D}^{+-}, \mathbb{D}^{-+}, \mathbb{D}^{--}$  such that

$$\begin{cases} \phi^{++}(\zeta)G^{++}(\zeta) + \phi^{--}(\zeta)G^{--}(\zeta) = g_1(\zeta), & \zeta \in \partial_0\mathbb{D}^2, \\ \phi^{+-}(\zeta)G^{+-}(\zeta) + \phi^{-+}(\zeta)G^{-+}(\zeta) = g_2(\zeta), & \zeta \in \partial_0\mathbb{D}^2. \end{cases} \quad (9)$$

Evidently adding these two equations one obtains Eq. (8) if one takes  $g_1(\zeta) + g_2(\zeta) = g(\zeta)$ . Moreover, by this formulation here we have one equation for every pair of analytic functions in reflective domains of the torus. Actually Eq. (9) is another equivalent formulation of Eq. (8) in the sense of pairs of domains  $(\mathbb{D}^{++}, \mathbb{D}^{--})$  and  $(\mathbb{D}^{-+}, \mathbb{D}^{+-})$ , if we take  $g_1$  as the relevant part of  $g$  for  $(\mathbb{D}^{++}, \mathbb{D}^{--})$  and  $g_2 = g - g_1$  for  $(\mathbb{D}^{-+}, \mathbb{D}^{+-})$ . Hence for  $(\mathbb{D}^{++}, \mathbb{D}^{--})$  the boundary values of  $(\phi^{++}, \phi^{--})$  provide no nontrivial value [1,20].

Again Eq. (9) can be transformed into the following:

$$\begin{cases} \phi^{++}(\zeta) + \phi^{--}(\zeta)G_1(\zeta) = g_1^0(\zeta), & \zeta \in \partial_0\mathbb{D}^2, \\ \phi^{+-}(\zeta) + \phi^{-+}(\zeta)G_2(\zeta) = g_2^0(\zeta), & \zeta \in \partial_0\mathbb{D}^2, \end{cases} \quad (10)$$

where

$$\begin{aligned} G_1(\zeta) &= G^{--}(\zeta)/G^{++}(\zeta) \neq 0, & g_1^0(\zeta) &= g_1(\zeta)/G^{++}(\zeta), \\ G_2(\zeta) &= G^{-+}(\zeta)/G^{+-}(\zeta) \neq 0, & g_2^0(\zeta) &= g_2(\zeta)/(G^{+-}(\zeta)). \end{aligned}$$

Formulation (10) is a much better one than (8). But (10) may not yet be a well-formulated problem. Go back to Eq. (1). That equation was established for one pair of functions  $\phi^+$  and  $\phi^-$ , analytic in  $\mathbb{D}^+$  and  $\mathbb{D}^-$ , respectively. Every one of the given functions  $G, g$  in (1) is the sum of boundary values of two functions, analytic in  $\mathbb{D}^+$  and  $\mathbb{D}^-$ , respectively, i.e.,  $G, g \in \partial\mathcal{H}^+ \oplus \partial\mathcal{H}^-$ .

The coefficients of (10) do not possess this property in general—they are sums of the boundary values of four analytic functions in respective domains, i.e.,  $G_1, G_2, g_1^0, g_2^0 \in \partial\mathcal{H}^{++} \oplus \partial\mathcal{H}^{--} \oplus \partial\mathcal{H}^{+-} \oplus \partial\mathcal{H}^{-+}$ . Suppose for simplicity  $g = 0$ . Taking log from both sides of (10) we have

$$\begin{cases} \log \phi^{++}(\zeta) - \log \phi^{--}(\zeta) = \log[-G_1(\zeta)], & \zeta \in \partial_0\mathbb{D}^2, \\ \log \phi^{+-}(\zeta) - \log \phi^{-+}(\zeta) = \log[-G_2(\zeta)], & \zeta \in \partial_0\mathbb{D}^2. \end{cases} \quad (11)$$

One can see that the left-hand side of Eq. (11) belongs to  $\partial\mathcal{H}^{++} \oplus \partial\mathcal{H}^{--}$  and  $\partial\mathcal{H}^{+-} \oplus \partial\mathcal{H}^{-+}$ , respectively. Thus in order Eq. (11) to be solvable conditions  $G_1 \in \partial\mathcal{H}^{++} \oplus \partial\mathcal{H}^{--}$  and  $G_2 \in \partial\mathcal{H}^{+-} \oplus \partial\mathcal{H}^{-+}$  have to be satisfied. Here we have some restrictions. Thus we see how to formulate a well-defined problem.

**Problem G<sub>2</sub>.** Let  $G, g \in W^2$  with  $G_1^0(\zeta), G_2^0(\zeta) \neq 0$ ,  $\zeta \in \partial_0 \mathbb{D}^2$ . Find analytic functions  $\phi^{++}, \phi^{+-}, \phi^{-+}, \phi^{--}$  in  $\mathbb{D}^{++}, \mathbb{D}^{+-}, \mathbb{D}^{-+}, \mathbb{D}^{--}$  such that

$$\begin{cases} \phi^{++}(\zeta) + \phi^{--}(\zeta)G_1^0(\zeta) = g_1^*(\zeta), & \zeta \in \partial_0 \mathbb{D}^2, \\ \phi^{+-}(\zeta) + \phi^{-+}(\zeta)G_2^0(\zeta) = g_2^*(\zeta), & \zeta \in \partial_0 \mathbb{D}^2, \end{cases} \quad (12)$$

where

$$\begin{aligned} G_1^0 &= G|_{\partial \mathcal{H}^{++} \oplus \partial \mathcal{H}^{--}}, & g_1^* &= g|_{\partial \mathcal{H}^{++} \oplus \partial \mathcal{H}^{--}}, \\ G_2^0 &= G|_{\partial \mathcal{H}^{+-} \oplus \partial \mathcal{H}^{-+}}, & g_2^* &= g|_{\partial \mathcal{H}^{+-} \oplus \partial \mathcal{H}^{-+}}. \end{aligned}$$

This is a well-posed problem. Introducing proper notations this formulation can be easily extended for any higher dimensional torus.

In [1,20] the division of the boundary values has been considered, the structure of the respective analytic functions and the torus related Cauchy integrals are defined.

Let the real-valued function  $\varphi$  belong to  $W(\partial_0 \mathbb{D}^2)$ . Then by the divided boundary values which are uniformly and absolutely convergent, respective analytic functions can be defined, see [1,20],

$$\begin{aligned} \varphi(\zeta_1, \zeta_2) &= \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} a_{k_1, k_2} \zeta_1^{k_1} \zeta_2^{k_2}, \\ a_{k_1, k_2} &= \frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \varphi(\eta_1, \eta_2) \eta_1^{-k_1} \eta_2^{-k_2} \frac{d\eta_1}{\eta_1} \frac{d\eta_2}{\eta_2}, \\ a_{-k_1, -k_2} &= \overline{a_{k_1, k_2}}, \quad k_1, k_2 \in \mathbb{Z}, \quad (\zeta_1, \zeta_2) \in \partial_0 \mathbb{D}^2, \\ \sum_{k_1=0}^{+\infty} \sum_{k_2=0}^{+\infty} a_{k_1, k_2} \zeta_1^{k_1} \zeta_2^{k_2} &= \frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \varphi(\zeta) \frac{\zeta}{\zeta - z} \frac{d\zeta}{\zeta} =: \phi^{++}(z), \quad z \in \mathbb{D}^2, \\ \sum_{\substack{k_1=0 \\ k_1+k_2>0}}^{+\infty} \sum_{k_2=0}^{+\infty} a_{-k_1, -k_2} \zeta_1^{-k_1} \zeta_2^{-k_2} &= \frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \varphi(\zeta) \left[ \frac{z}{z - \zeta} - 1 \right] \frac{d\zeta}{\zeta} \\ &=: \phi^{--}(z), \quad z \in \mathbb{D}^{-2}, \\ \sum_{k_1=1}^{+\infty} \sum_{k_2=1}^{+\infty} a_{k_1, -k_2} \zeta_1^{k_1} \zeta_2^{-k_2} &= \frac{-1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \varphi(\zeta) \frac{z_1}{\zeta_1 - z_1} \frac{z_2}{\zeta_2 - z_2} \frac{d\zeta}{\zeta} \\ &=: -\phi^{+-}(z), \quad z \in \mathbb{D}^+ \times \mathbb{D}^-, \\ \sum_{k_1=1}^{+\infty} \sum_{k_2=1}^{+\infty} a_{-k_1, k_2} \zeta_1^{-k_1} \zeta_2^{k_2} &= \frac{-1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \varphi(\zeta) \frac{\zeta_1}{\zeta_1 - z_1} \frac{z_2}{\zeta_2 - z_2} \frac{d\zeta}{\zeta} \\ &=: -\phi^{-+}(z), \quad z \in \mathbb{D}^- \times \mathbb{D}^+. \end{aligned} \quad (13)$$

Evidently, the Cauchy kernels for every analytic function differ from each other substantially. Therefore, making the definition of a common Cauchy integral as starting point to discuss the torus related inhomogeneous Riemann problem, as it was done in [5], does not look very proper.



Of course only in the one variable case the Cauchy kernel is the same for  $\phi^+(z)$  and  $\phi^-(z)$  as well.

Due to  $\varphi \in W(\partial_0\mathbb{D}^2, \mathbb{R})$  in (13) the analytic functions  $\phi^{++}, \phi^{--}, \phi^{+-}, \phi^{-+}$  are also uniformly and absolutely convergent even on  $\partial_0\mathbb{D}^2$ . Taking this fact into account, for complex-valued  $\varphi$  instead of a real one in (13), we have

$$\phi^{++}(\eta) - \phi^{+-}(\eta) - \phi^{-+}(\eta) + \phi^{--}(\eta) = \varphi(\eta), \quad \eta \in \partial_0\mathbb{D}^2, \quad (14)$$

or equivalently

$$\begin{cases} \phi^{++}(\eta) + \phi^{--}(\eta) = \varphi^{++}(\eta) + \varphi^{--}(\eta), & \eta \in \partial_0\mathbb{D}^2, \\ -\phi^{+-}(\eta) - \phi^{-+}(\eta) = \varphi^{+-}(\eta) + \varphi^{-+}(\eta), & \eta \in \partial_0\mathbb{D}^2, \end{cases} \quad (15)$$

where

$$\begin{aligned} \phi^{++}(\eta) + \varphi^{--}(\eta) &\in \partial\mathcal{H}^{++} \oplus \partial\mathcal{H}^{--}, & \varphi^{+-}(\eta) + \varphi^{-+}(\eta) &\in \partial\mathcal{H}^{+-} \oplus \partial\mathcal{H}^{-+}, \\ \phi^{++}(\eta) + \varphi^{--}(\eta) + \varphi^{+-}(\eta) + \varphi^{-+}(\eta) &= \varphi(\eta), & \zeta &\in \partial_0\mathbb{D}^2. \end{aligned}$$

Define

$$\phi(z) := \frac{1}{(2\pi i)^2} \int_{\partial_0\mathbb{D}^2} \varphi(\zeta) C(\zeta, z) \frac{d\zeta}{\zeta}, \quad (16)$$

where

$$C(\zeta, z) = \begin{cases} \frac{\zeta}{\zeta - z}, & z \in \mathbb{D}^2, \\ \frac{z_1}{\zeta_1 - z_1} \frac{\zeta_2}{\zeta_2 - z_2}, & (z_1, z_2) \in \mathbb{D}^{+-}, \\ \frac{\zeta_1}{\zeta_1 - z_1} \frac{z_2}{\zeta_2 - z_2}, & (z_1, z_2) \in \mathbb{D}^{-+}, \\ \left[ \frac{z}{\zeta - \zeta} - 1 \right], & z \in \mathbb{D}^{--}, \\ \frac{\zeta}{\zeta - z}, & z \in \partial_0\mathbb{D}^2. \end{cases}$$

Since  $\varphi \in W^2$ , the function  $\phi(\eta)$  on  $\partial_0\mathbb{D}^2$  exists as a Fourier series which is absolutely and uniformly convergent. Applying (13) to (16) for  $\eta \in \partial_0\mathbb{D}^2$  we have

$$\begin{aligned} \phi(\eta) &= \frac{1}{(2\pi i)^2} \int_{\partial_0\mathbb{D}^2} \varphi(\zeta) \frac{d\zeta}{\zeta - \eta} \\ &= \frac{1}{(2\pi i)^2} \int_{\partial_0\mathbb{D}^2} \left[ \sum_{k_1=1}^{+\infty} \zeta_1^{k_1} + 1 \right] \left[ \sum_{k_2=1}^{+\infty} \zeta_2^{k_2} + 1 \right] a_{k_1, k_2} \frac{d\zeta}{\zeta - \eta} \\ &\quad + \frac{1}{(2\pi i)^2} \int_{\partial_0\mathbb{D}^2} \left[ \left( \sum_{k_1=1}^{+\infty} \zeta_1^{-k_1} + 1 \right) \left( \sum_{k_2=1}^{+\infty} \zeta_2^{-k_2} + 1 \right) - 1 \right] a_{-k_1, -k_2} \frac{d\zeta}{\zeta - \eta} \\ &\quad + \frac{1}{(2\pi i)^2} \int_{\partial_0\mathbb{D}^2} \left[ \sum_{k_1=1}^{+\infty} \zeta_1^{k_1} \sum_{k_2=1}^{+\infty} \zeta_2^{-k_2} a_{k_1, -k_2} + \sum_{k_1=1}^{+\infty} \zeta_1^{-k_1} \sum_{k_2=1}^{+\infty} \zeta_2^{k_2} a_{-k_1, k_2} \right] \frac{d\zeta}{\zeta - \eta} \\ &= \frac{1}{2^2} \left[ \sum_{k_1=1}^{+\infty} \eta_1^{k_1} + 1 \right] \left[ \sum_{k_2=1}^{+\infty} \eta_2^{k_2} + 1 \right] a_{k_1, k_2} \end{aligned}$$

$$+ \frac{1}{2^2} \left[ \left( \sum_{k_1=1}^{+\infty} \eta_1^{-k_1} - 1 \right) \left( \sum_{k_2=1}^{+\infty} \eta_2^{-k_2} - 1 \right) - 1 \right] a_{-k_1, -k_2} \\ - \frac{1}{2^2} \left[ \sum_{k_1=1}^{+\infty} \eta_1^{k_1} \sum_{k_2=1}^{+\infty} \eta_2^{-k_2} a_{k_1, -k_2} + \sum_{k_1=1}^{+\infty} \eta_1^{-k_1} \sum_{k_2=1}^{+\infty} \eta_2^{k_2} a_{-k_1, k_2} \right],$$

i.e., the first, the third and the fourth terms are  $\phi^{++}(\eta)$ ,  $\phi^{+-}(\eta)$  and  $\phi^{-+}(\eta)$ . But the second term is not equal to  $\phi^{--}(\eta)$ . So for  $\partial_0 \mathbb{D}^2$  we do not have an analogue of (9). Why could not we get such a formula? By the above Cauchy integral of  $\varphi$  on  $\partial_0 \mathbb{D}^2$  we see that

$$\phi(\eta) = \frac{1}{2^2} [\phi_*^{++}(\eta) + \phi_*^{+-}(\eta) + \phi_*^{-+}(\eta) + \phi_*^{--}(\eta)], \quad \eta \in \partial_0 \mathbb{D}^2, \quad (17)$$

where

$$\phi_*^{++}(\eta) = \frac{1}{(\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \phi^{++}(\zeta) \frac{d\zeta}{\zeta - \eta} = \phi^{++}(\eta),$$

$$\phi_*^{+-}(\eta) = \frac{-1}{(\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \phi^{+-}(\zeta) \frac{d\zeta}{\zeta - \eta} = \phi^{+-}(\eta),$$

$$\phi_*^{-+}(\eta) = \frac{-1}{(\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \phi^{-+}(\zeta) \frac{d\zeta}{\zeta - \eta} = \phi^{-+}(\eta),$$

$$\phi_*^{--}(\eta) = \frac{1}{(\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \phi^{--}(\zeta) \frac{d\zeta}{\zeta - \eta} \neq \phi^{--}(\eta)$$

and

$$\phi^{++}(\zeta) = \lim_{\substack{z \rightarrow \zeta \in \partial_0 \mathbb{D}^2 \\ z \in D^{++}}} \phi(z), \quad \phi^{--}(\zeta) = \lim_{\substack{z \rightarrow \zeta \in \partial_0 \mathbb{D}^2 \\ z \in D^{--}}} \phi(z),$$

$$\phi^{-+}(\zeta) = \lim_{\substack{z \rightarrow \zeta \in \partial_0 \mathbb{D}^2 \\ z \in D^{-+}}} \phi(z), \quad \phi^{+-}(\zeta) = \lim_{\substack{z \rightarrow \zeta \in \partial_0 \mathbb{D}^2 \\ z \in D^{+-}}} \phi(z).$$

We call  $\phi_*^{++}(\eta)$ ,  $\phi_*^{--}(\eta)$ ,  $\phi_*^{+-}(\eta)$ ,  $\phi_*^{-+}(\eta)$  the boundary integral conjugate of  $\phi^{++}(\eta)$ ,  $\phi^{--}(\eta)$ ,  $\phi^{+-}(\eta)$ ,  $\phi^{-+}(\eta)$ ,  $\eta \in \partial_0 \mathbb{D}^2$ , respectively.

Interestingly for  $\eta \in \partial_0 \mathbb{D}^2$  we have (also for the other three pairs of functions)

$$\frac{1}{(\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \phi^{--}(\zeta) \frac{d\zeta}{\zeta - \eta} = \phi_*^{--}(\eta), \quad \frac{1}{(\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \phi_*^{--}(\zeta) \frac{d\zeta}{\zeta - \eta} = \phi^{--}(\eta). \quad (18)$$

Although the formula (14) and the formula (15) are equivalent, having the formula (14) in the form of (15) has more advantage: (15) is about two pluriharmonic functions in  $\mathbb{D}^2$  ( $\mathbb{D}^{-2}$ ) and  $\mathbb{D}^{+-}$  ( $\mathbb{D}^{-+}$ ). If the right-hand side is real, then it can be reduced to two Schwarz problems for analytic functions in  $\mathbb{D}^2$  ( $\mathbb{D}^{-2}$ ) and  $\mathbb{D}^{+-}$  ( $\mathbb{D}^{-+}$ ). The formula (17) does not hold for the original analytic functions in the two-dimensional space but for their boundary integral conjugates. Actually this is not strange at all. Because in the one variable case we have

$$\phi_*^+(\eta) = \frac{1}{\pi i} \int_{\partial \mathbb{D}} \phi^+(\zeta) \frac{d\zeta}{\zeta - \eta} = \phi^+(\eta), \quad \phi_*^-(\eta) = \frac{-1}{\pi i} \int_{\partial \mathbb{D}} \phi^-(\zeta) \frac{d\zeta}{\zeta - \eta} = \phi^-(\eta), \quad (19)$$

therefore we can replace (7) with

$$2\phi(\zeta) = \phi_*^+(\zeta) + \phi_*^-(\zeta), \quad \zeta \in \partial\mathbb{D}. \quad (20)$$

We see that the conjugates and the original ones are identical. Now we see a different perspective of the Plemelj–Sokhotzki formula: on one hand it is about the series representation and on the other about the integral of the given functions on the boundary.

However, the relationship (18) can reveal enough information to determine the original ones. There are two ways to formulate the Riemann problem in the higher dimensional space: in the form (8) and in the equivalent form of (9) or (12).

We study the problem on the torus but just not on the whole boundary, i.e., the form (8) has to be reduced to the form (9) or (12). For (8) there is no inside and outside in the sense of the torus, but for (9) or (12) there is. If (9) or (12) is solvable, undoubtedly (8) is solvable. So splitting the form (8) into (9) or (12) is very comprehensive. We can even formulate the problem directly in the form of (9) or (12). At least it is sure that if we could solve the problem (9) or (12) the problem (8) is always solvable.

Clearly

$$\begin{aligned} \phi^{+-}(0, z_2) &= 0, \quad \phi^{+-}(z_1, \infty) = 0, \quad |z_1| \leq 1, \quad |z_2| \geq 1; \\ \phi^{-+}(\infty, \eta_2) &= 0, \quad \phi^{-+}(\eta_1, 0) = 0, \quad |\eta_1| \geq 1, \quad |\eta_2| \leq 1, \quad \phi^{--}(\infty, \infty) = 0. \end{aligned} \quad (21)$$

But  $\phi^{--}(z_1, \infty)$  and  $\phi^{--}(\infty, z_2)$  are not necessarily zero as it was needed in [11], except at most  $\phi^{--}(\infty, \infty) = 0$ .

It is known that for  $\eta \in \partial\mathbb{D}$ ,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \zeta^k \frac{d\zeta}{\zeta - \eta} &= \frac{1}{2} \eta^k, \quad \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \zeta^{-h} \frac{d\zeta}{\zeta - \eta} = -\frac{1}{2} \eta^{-h}, \\ k &\in \mathbb{N} \cup \{0\} =: \mathbb{Z}_+, \quad h \in \mathbb{N}. \end{aligned} \quad (22)$$

Although simple checking methods for holomorphic functions (22) worked well in [5] due to their unanimous form of the Cauchy kernel, to our case, due to the varying form of the Cauchy kernel, it is useless. So we have to find another checking method.

In [20] a simple checking method for boundary values of holomorphic functions in any torus domains is given. In order to know whether  $\phi^- \in \partial\mathcal{H}^-$  in one variable case it is enough to know if  $\bar{\phi}^- \in \partial\mathcal{H}^+$  satisfying  $\phi^-(\infty) = 0$ . This idea can be applied to check boundary values of holomorphic functions in arbitrary torus domains. However, we need to introduce a slightly modified version of a complex conjugate.

As we have seen the given function  $\varphi \in W(\partial_0\mathbb{D}^2, \mathbb{C})$  in (13) can be split into four respective parts  $\phi^{++}, \phi^{--}, \phi^{+-}, \phi^{-+}$  so that  $\phi^{++} \in \partial\mathcal{H}^{++}, \phi^{--} \in \partial\mathcal{H}^{--}, \phi^{+-} \in \partial\mathcal{H}^{+-}, \phi^{-+} \in \partial\mathcal{H}^{-+}$ . In [1,20] some equivalent methods are provided for a given function to check if the function values are boundary values of an analytic function in a polydisc. For the boundary values of analytic functions in the other torus domains the concept of *partial conjugate* is introduced. Thus  $\phi^{--} \in \partial\mathcal{H}^{--}, \phi^{+-} \in \partial\mathcal{H}^{+-}, \phi^{-+} \in \partial\mathcal{H}^{-+}$  is equivalent to  $\mathcal{C}_\zeta[\phi^{--}] = \bar{\phi}^{--}, \mathcal{C}_{\zeta_2}[\phi^{+-}], \mathcal{C}_{\zeta_1}[\phi^{-+}] \in \partial\mathcal{H}^{++}$  satisfying condition (21) in respective domains.

So far we have established the Plemelj–Sokhotzki formula and the well-posed formulation of the Riemann problem in the two variable case. Now we can move to the higher dimensional case. Although one can write the equation in the form of (8) in  $\mathbb{C}^2$  it would be very inconvenient in higher dimensional spaces. One has to find a better way of description. For this reason we introduce the following notation, see also [20].

**Definition.** Let  $\chi = (\chi_1, \dots, \chi_n)$  be a multi-sign, where

$$\begin{aligned} \chi_1, \dots, \chi_n \in \{+, -\}, \quad 0 \leq v \leq n, \quad 1 \leq \sigma_1 < \dots < \sigma_v \leq n, \\ 1 \leq \sigma_{v+1} < \dots < \sigma_n \leq n, \quad \{\sigma_1, \dots, \sigma_n\} = \{1, \dots, n\}, \quad \chi_{\sigma_1} = -, \dots, \chi_{\sigma_v} = -, \\ \chi_{\sigma_{v+1}} = +, \dots, \chi_{\sigma_n} = +, \quad \chi(v) = \chi_{\sigma_1 \dots \sigma_v}(v), \end{aligned}$$

where  $v$  gives the number of minus  $(-)$  signs and the indices  $\sigma_1, \dots, \sigma_v$  show the position of these minus sign components.  $\chi(v)$  obviously has  $(n - v)$  plus  $(+)$  sign components at the positions  $\sigma_{v+1}, \dots, \sigma_n$ . In addition  $\chi(v) = \chi_{\sigma_1 \dots \sigma_v}(v) = -\chi_{\rho_1 \dots \rho_{n-v}}(n - v) = -\chi(n - v)$ , for  $0 \leq v \leq n$  and  $\{\rho_1, \dots, \rho_{n-v}\} = \{1, \dots, n\} \setminus \{\sigma_1, \dots, \sigma_v\} = \{\sigma_{v+1}, \dots, \sigma_n\}$ , when treating  $\chi(v)$  as a vector.

For convenience we denote  $\mathbb{D}_{\sigma_1}^- \times \dots \times \mathbb{D}_{\sigma_v}^- \times \mathbb{D}_{\sigma_{v+1}}^+ \times \dots \times \mathbb{D}_{\sigma_n}^+$  as  $\mathbb{D}^{\chi_{\sigma_1 \dots \sigma_v}(v)}$  and  $\mathbb{D}_{\sigma_1}^+ \times \dots \times \mathbb{D}_{\sigma_v}^+ \times \mathbb{D}_{\sigma_{v+1}}^- \times \dots \times \mathbb{D}_{\sigma_n}^-$  as  $\mathbb{D}^{-\chi_{\sigma_1 \dots \sigma_v}(v)}$ .

Actually  $\chi_{\sigma_1 \dots \sigma_v}(v)$ ,  $0 \leq v \leq n$ , can be understood as signs of vertices of the  $n$ -dimensional cube  $[-1, +1]^n$ . In the case  $n = 2$  the signs  $(+, +)$ ,  $(+, -)$ ,  $(-, +)$ ,  $(-, -)$  correspond to the signs of the vertices  $(1, 1)$ ,  $(1, -1)$ ,  $(-1, 1)$ ,  $(-1, -1)$  of the unit square. Therefore we denote  $\chi^*$  as the vertices of the  $[-1, +1]^n$  cube, while  $\chi$  represents the respective multi-sign.

Let the real-valued  $\varphi$  belong to  $W(\partial_0 \mathbb{D}^n, \mathbb{C})$ . Then  $\varphi$  can be represented as

$$\varphi(\eta) = \sum_{\kappa \in \mathbb{Z}^n} \alpha_\kappa \eta^\kappa, \quad \alpha_\kappa = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \varphi(\zeta) \zeta^{-\kappa} \frac{d\zeta}{\zeta}, \quad \bar{\alpha}_\kappa = \alpha_{-\kappa}, \quad \kappa \in \mathbb{Z}^n. \quad (23)$$

This Fourier series is absolutely and uniformly convergent to  $\varphi(\eta)$ ,  $\eta \in \partial_0 \mathbb{D}^n$ , because of  $\varphi \in W(\partial_0 \mathbb{D}^n, \mathbb{C})$  and it can be split into  $2^n$  parts:

$$\left\{ \begin{aligned} & \left[ \prod_{t=1}^n \left( \sum_{k_t=1}^{+\infty} \zeta_t^{-k_t} + 1 \right) - 1 \right] \alpha_{-k_1, \dots, -k_n} \\ &= \sum_{|\kappa| > 0, \kappa \in \mathbb{Z}_+^n} \alpha_{-\kappa} \zeta^{-\kappa} =: (-1)^n \phi^{\chi(n)}(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n, \\ & \prod_{t=1}^n \left( \sum_{k_t=1}^{+\infty} (\zeta_t^{\chi_t^*})^{k_t} + \delta_t^\chi \right) \alpha_{\chi_1^* k_1, \dots, \chi_n^* k_n} =: (-1)^v \phi^{\chi(v)}(\zeta), \quad 0 \leq v < n, \zeta \in \partial_0 \mathbb{D}^n, \end{aligned} \right. \quad (24)$$

where

$$\delta_t^{\chi_t} = \frac{|\chi_t^* + \chi_{t^*+1}^*|}{2}, \quad 1 \leq t \leq n, \quad t^* = t \bmod (n).$$

Every  $\phi^{\chi(v)}(\zeta)$  is the boundary value of a holomorphic function  $\phi^{\chi(v)}(z)$  in  $\mathbb{D}^{\chi(v)}$ . Denote

$$\left\{ \begin{aligned} & \left[ \prod_{t=1}^n \left( \sum_{k_t=1}^{+\infty} \zeta_t^{-k_t} - 1 \right) - (-1)^n \right] \alpha_{-k_1, \dots, -k_n} =: \phi_*^{\chi(n)}(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n, \\ & \prod_{t=1}^v \left( \sum_{k_{\sigma_t}=1}^{+\infty} \zeta_{\sigma_t}^{-k_{\sigma_t}} - \delta_{\sigma_t}^- \right) \prod_{t=v+1}^n \left( \sum_{k_{\sigma_t}=1}^{+\infty} \zeta_{\sigma_t}^{k_{\sigma_t}} + \delta_{\sigma_t}^+ \right) \alpha_{\chi_1^* k_1, \dots, \chi_n^* k_n} =: \phi_*^{\chi(v)}(\zeta), \\ & 0 \leq v < n, \zeta \in \partial_0 \mathbb{D}^n, \end{aligned} \right. \quad (25)$$

as boundary integral conjugates of (24). Evidently for  $\eta \in \partial_0 \mathbb{D}^n$ ,

$$\frac{1}{(\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \phi^{\chi^{(v)}}(\zeta) \frac{d\zeta}{\zeta - \eta} = \phi_*^{\chi^{(v)}}(\eta), \quad \frac{1}{(\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \phi_*^{\chi^{(v)}}(\zeta) \frac{d\zeta}{\zeta - \eta} = \phi^{\chi^{(v)}}(\eta). \quad (26)$$

Correctness of the partition of (23) as (24) can be proven by the following lemma, see [1,20].

**Lemma 1.**

$$\begin{aligned} \prod_{t=1}^n (a_t + \bar{a}_t + 1) + 1 &= \prod_{t=1}^n [(a_t + \delta_t^+) + (\bar{a}_t + \delta_t^-)] \\ &= \sum_{v=0}^n \sum_{\substack{1 \leq \sigma_1 < \dots < \sigma_v \leq n \\ 1 \leq \sigma_{v+1} < \dots < \sigma_n \leq n}} \prod_{t=1}^v (\bar{a}_{\sigma_t} + \delta_{\sigma_t}^-) \prod_{t=v+1}^n (a_{\sigma_t} + \delta_{\sigma_t}^+) \end{aligned}$$

for  $a_t \in \mathbb{C}$ ,  $1 \leq t \leq n$ , where

$$\begin{aligned} \text{cd}\{\sigma_1, \dots, \sigma_v, \sigma_{v+1}, \dots, \sigma_n\} &= n, \quad \chi_{\sigma_1}^* = \dots = \chi_{\sigma_v}^* = -1, \\ \chi_{\sigma_{v+1}}^* &= \dots = \chi_{\sigma_n}^* = +1. \end{aligned}$$

If the  $\delta_k$  are treated as numbers, then there is an interesting fact:

$$\begin{aligned} 1 &= \frac{|\chi_{t+1}^* + 1|}{2} + \frac{|\chi_{t+1}^* - 1|}{2} =: \delta_t^+ + \delta_t^-, \quad 1 \leq t \leq n-1, \\ 1 &= \frac{|\chi_1^* + 1|}{2} + \frac{|\chi_1^* - 1|}{2} =: \delta_n^+ + \delta_n^-. \end{aligned}$$

However, throughout our paper we interpret  $\delta_h$  ( $1 \leq h \leq n$ ) as components of an  $n$ -dimensional tuple. Any element of the tuple is composed of exactly just  $n$  components, including some  $\delta_k$  or  $a_t$  ( $k+t=n$ ,  $0 \leq k, t \leq n$ ). The dimension of this kind of tuples  $n$  must satisfy  $n \geq 2$ . Any of  $\delta_k$  alone does not make any sense, unless it comes with the other  $n-1$  components together with an element of the set of tuples.

By the definition of boundary values (24), an arbitrary analytic function  $\phi^{\chi^{(v)}}(z)$  in  $\mathbb{D}^{\chi^{(v)}}$  with boundary values in  $W^n$  and continuous on  $\partial_0 \mathbb{D}^n$ , without loss of generality, possesses the form

$$\begin{aligned} &\left\{ \prod_{t=1}^n \left( \sum_{k_t=1}^{+\infty} (z_t^{k_t})^{\chi_t^*} + \delta_t^{\chi_t^*} \right) \alpha_{\chi_1^* k_1, \dots, \chi_n^* k_n} =: (-1)^v \phi^{\chi^{(v)}}(z), \quad 0 \leq v < n, \quad z \in \mathbb{D}^{\chi^{(v)}}, \right. \\ &\left. \left[ \prod_{t=1}^n \left( \sum_{k_t=1}^{+\infty} z_t^{-k_t} + 1 \right) - 1 \right] \alpha_{-k_1, \dots, -k_n} =: (-1)^n \phi^{\chi^{(n)}}(z), \quad z \in \mathbb{D}^{-n}, \right. \end{aligned} \quad (27)$$

and they converge absolutely and uniformly even on  $\partial_0 \mathbb{D}^n$ .

After having the boundary values of holomorphic functions classified and the respective Cauchy kernels established in  $W(\partial_0 \mathbb{D}^n, \mathbb{C})$ , we define respective Cauchy integrals in  $C^\alpha(\partial_0 \mathbb{D}^n, \mathbb{C})$ ,  $0 < \alpha < 1$ , not restricting our discussion to  $W(\partial_0 \mathbb{D}^n, \mathbb{C})$ .

We define for  $\varphi \in C^\alpha(\partial_0 \mathbb{D}^n, \mathbb{C})$ ,  $0 < \alpha < 1$ , an integral

$$\phi(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \varphi(\zeta) C(\zeta, z) \frac{d\zeta}{\zeta}, \quad (28)$$

where

$$C(\zeta, z) = \begin{cases} (-1)^v \prod_{k=1}^n \left[ \frac{(z_k \zeta_k^{-1})^{\chi_k^*}}{1 - (z_k \zeta_k^{-1})^{\chi_k^*}} + \delta_k^\chi \right], & 0 \leq v \leq n-1, \quad z \in \mathbb{D}^{\chi(v)}, \\ (-1)^n \left[ \frac{z}{z-\zeta} - 1 \right], & v = n, \quad z \in \mathbb{D}^{-n}, \\ \frac{\zeta}{\zeta-z}, & z \in \partial_0 \mathbb{D}^n. \end{cases}$$

Obviously by the division of the boundary values (24) all the related analytic functions can be represented as (28). We call (28) *the torus related Cauchy integral*.

The function  $\phi^{\chi(v)}(z)$  defined by (28) has the following property.

Let  $k$  be a fixed integer in  $\{1, \dots, n\}$ ,  $\chi(v)$  be a fixed sign and  $z \in \mathbb{D}^{\chi(v)}$ .

$$\begin{cases} \phi^{\chi(v)}(z)|_{z_{k^*}=\infty} = 0, & \text{for } k^* \in \{\sigma_1, \dots, \sigma_v\} \text{ and } k^* + 1 \in \{\sigma_{v+1}, \dots, \sigma_n\}, \\ \phi^{\chi(v)}(z)|_{z_{k^*}=0} = 0, & \text{for } k^* \in \{\sigma_{v+1}, \dots, \sigma_n\} \text{ and } k^* + 1 \in \{\sigma_1, \dots, \sigma_v\}. \end{cases} \quad (29)$$

The analytic functions defined by (27) can be obtained from (28) in the respective domains and their boundary values (24) can also be given by

$$\phi^{\chi(v)}(\zeta) := \lim_{\substack{z \rightarrow \zeta \in \partial_0 \mathbb{D}^n \\ z \in \mathbb{D}^{\chi(v)}}} \phi(z).$$

Interestingly

$$\overline{\prod_{t=1}^n (a_t^{\chi_t} + \delta_t^\chi)} = \prod_{t=1}^n (a_t^{-\chi_t} + \delta_t^\chi)$$

so if  $\varphi(\eta)$  is real and  $\varphi(0) = 0$ , then

$$\begin{aligned} \overline{(-1)^v \phi^{\chi(v)}(\zeta)} &:= \overline{\prod_{t=1}^n \left( \sum_{k_t=1}^{+\infty} (\zeta_t^{k_t})^{\chi_t^*} + \delta_t^\chi \right) \alpha_{\chi_1^* k_1, \dots, \chi_n^* k_n}} \\ &= \prod_{t=1}^n \left( \sum_{k_t=1}^{+\infty} (\zeta_t^{k_t})^{-\chi_t^*} + \delta_t^\chi \right) \alpha_{-\chi_1^* k_1, \dots, -\chi_n^* k_n} \\ &= (-1)^v \phi^{-\chi(v)}(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n, \quad 0 \leq v \leq n, \end{aligned}$$

holds and  $\phi^{\chi(v)}(\zeta)$  can be seen as the reflection of  $\phi^{-\chi(v)}(\zeta)$  with respect to  $\partial_0 \mathbb{D}^n$ . This property of the boundary values will be quite useful in our further discussions.

Paying attention to (23) and (24), using Lemma 1 and applying (22) to (28) and taking the definition of the boundary integral conjugate into account the next result is evident.

**Theorem 2** (Plemelj–Sokhotzki formula for torus domains). *Under the condition  $\varphi \in C^\alpha(\partial_0 \mathbb{D}^n, \mathbb{C})$  with  $0 < \alpha < 1$ , the boundary values of the function  $\phi^{\chi(v)}$  which is holomorphic in  $\mathbb{D}^{\chi(v)}$  and defined as in (28) satisfy*

$$(-1)^v \phi^{\chi(v)}(\zeta) + (-1)^{n-v} \phi^{-\chi(v)}(\zeta) = \varphi^{\chi(v)}(\zeta) + \varphi^{-\chi(v)}(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n, \quad (30)$$

$$2^n \phi(\zeta) = \sum_{\chi(v)} \phi_*^{\chi(v)}(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n, \quad (31)$$

where  $\varphi^{\chi(v)}(\zeta) \in \partial \mathcal{H}^{\chi(v)}$ , can be given by Fourier series for  $\varphi(\zeta)$  on  $\partial_0 \mathbb{D}^n$ , i.e.,

$$\varphi^{\chi(v)}(\zeta) + \varphi^{-\chi(v)}(\zeta) = \varphi(\zeta)|_{\partial \mathcal{H}^{\chi(v)} \oplus \partial \mathcal{H}^{-\chi(v)}}, \quad \zeta \in \partial_0 \mathbb{D}^n.$$

Further  $\phi_*^{\chi(v)}(\zeta)$  is the boundary integral conjugate of  $\phi^{\chi(v)}(\zeta)$  defined in (25). The summation over  $\chi(v)$  actually runs over all  $\sigma$ s, see Lemma 1.

In order to know if a given function is the boundary value of a holomorphic function in  $\mathbb{D}^{\chi(v)}$  we can apply a simple method. Therefore we introduce a simple concept.

We define the *partial complex conjugate* of  $\phi^{\chi(v)}$  as

$$\begin{aligned} C_{\zeta}[\phi^{\chi(n)}(\zeta)] &:= \overline{[\phi^{\chi(n)}(\zeta)]_{\zeta}} := \left[ \prod_{t=1}^n \left( 1 + \sum_{k_t=1}^{+\infty} \zeta_t^{-k_t} \right) \bar{\alpha}_{-k_1, \dots, -k_n} - \bar{\alpha}_{0, \dots, 0} \right] \\ &= \left[ \prod_{t=1}^n \left( 1 + \sum_{k_t=1}^{+\infty} \zeta_t^{k_t} \right) \bar{\alpha}_{-k_1, \dots, -k_n} - \bar{\alpha}_{0, \dots, 0} \right], \quad \zeta \in \partial_0 \mathbb{D}^n, \end{aligned} \quad (32)$$

$$\begin{aligned} C_{\zeta_{\sigma_1} \dots \zeta_{\sigma_v}}[\phi^{\chi(v)}(\zeta)] &:= \overline{[\phi^{\chi(v)}(\zeta)]_{\zeta_{\sigma_1} \dots \zeta_{\sigma_v}}} \\ &:= \left[ \prod_{t=1}^v \left( \delta_{\sigma_t}^- + \sum_{k_{\sigma_t}=1}^{+\infty} \zeta_{\sigma_t}^{-k_{\sigma_t}} \right) \prod_{t=v+1}^n \left( \delta_{\sigma_t}^+ + \sum_{k_{\sigma_t}=1}^{+\infty} \zeta_{\sigma_t}^{k_{\sigma_t}} \right) \alpha_{\chi_1^* k_1, \dots, \chi_n^* k_n} \right] \\ &= \left[ \prod_{t=1}^n \left( \delta_t^{\chi_t} + \sum_{k_t=1}^{+\infty} \zeta_t^{k_t} \right) \alpha_{\chi_1^* k_1, \dots, \chi_n^* k_n} \right], \quad 0 < v < n, \quad \zeta \in \partial_0 \mathbb{D}^n. \end{aligned} \quad (33)$$

Obviously  $\phi^{\chi(v)}(\zeta) \in \partial \mathcal{H}(\mathbb{D}^{\chi(v)})$  ( $0 < v \leq n$ ) is equivalent to  $C_{\zeta_{\sigma_1} \dots \zeta_{\sigma_v}}[\phi^{\chi(v)}(\zeta)] \in \partial \mathcal{H}(\mathbb{D}^n)$  ( $0 < v \leq n$ ) if condition (29) is satisfied. Thus we have

**Theorem 3.** Let  $\phi^{\chi(v)} \in C^{\alpha}(\partial_0 \mathbb{D}^n, \mathbb{C})$  with  $0 < \alpha < 1$  and continuous in  $\mathbb{D}^{\chi(v)}$ . Suppose  $\phi^{\chi(v)}$  satisfies condition (29) and  $\phi^{\chi(n)}(\infty) = 0$  for  $v = n$ . Then  $C_{\zeta_{\sigma_1} \dots \zeta_{\sigma_v}}[\phi^{\chi(v)}(\zeta)] \in \partial \mathcal{H}(\mathbb{D}^n)$  is the necessary and sufficient condition for  $\phi^{\chi(v)}(\zeta) \in \partial \mathcal{H}(\mathbb{D}^{\chi(v)})$ .

Now we come to the formulation of the problem. Although there was no truly  $n(n > 2)$ -dimensional consideration of the problem, from the former investigations one can see that the formulation of the Riemann problem in torus related domains generally had the following form.

**Problem  $R_n$  ( $n \geq 1$ ).** Let  $G, g \in C^{\alpha}(\partial_0 \mathbb{D}^n, \mathbb{C})$ ,  $0 < \alpha < 1$ . Find analytic functions  $\phi^{\chi(v)}(\zeta)$  in  $\mathbb{D}^{\chi(v)}$ ,  $0 \leq v \leq n$ , such that

$$\sum_{\chi(v)} G^{\chi(v)}(\zeta) \phi^{\chi(v)}(\zeta) = g(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n, \quad (34)$$

where  $G^{\chi(v)}(\zeta) = G(\zeta)|_{\partial \mathcal{H}^{\chi(v)} \oplus \partial \mathcal{H}^{-\chi(v)}}$  with  $G^{\chi(v)}(\zeta) \neq 0$ ,  $\zeta \in \partial_0 \mathbb{D}^n$ .

According to [1,20] the whole space  $\mathbb{C}^n$  is split into  $2^{n-1}$  pairs of different polydomains by the torus. For every pair, one domain is the reflection of its counterpart domain through the torus, one can be considered as the inner domain and the other as its outer domain. A given function on the torus can be split uniquely into boundary values of analytic functions in the respective domains. Boundary values of an analytic function in one polydomain have at most some effect on the boundary values of analytic functions in its reflection domain, i.e., any domain  $\mathbb{D}^{\chi(v)}$  has

no interactions with the other  $\mathbb{D}^{\chi(\mu)}$  ( $0 \leq \mu \leq n$ ,  $\mu \neq v$ ,  $n - v$ ), except its own reflection  $\mathbb{D}^{-\chi(v)}$  through  $\partial_0 \mathbb{D}^n$ . This means Eq. (34) is actually a sum of  $2^{n-1}$  single equations. Every single equation deals with boundary values of a pair of analytic functions in a respective domain pair. Therefore Eq. (34) actually can be split into

$$\begin{aligned} & (-1)^v \phi^{\chi(v)}(\zeta) G^{\chi(v)}(\zeta) + G^{-\chi(v)}(\zeta) \phi^{-\chi(v)}(\zeta) \\ & = g^{\chi(v)}(\zeta) + g^{-\chi(v)}(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n, \end{aligned} \quad (35)$$

where  $g^{\chi(v)} + g^{-\chi(v)}$  is defined by the Fourier series of  $g$  on  $\partial_0 \mathbb{D}^n$ , i.e.,  $g^{\chi(v)} + g^{-\chi(v)} = g|_{\partial \mathcal{H}^{\chi(v)} \oplus \partial \mathcal{H}^{-\chi(v)}}$ .

The viewpoint or subject of this formulation is always a *single pair of analytic functions or a single pair of torus domains* but not a *single variable* as it was done in [2,12].

However, the formulation (35) might not be well posed (some solvability conditions must be satisfied, if  $G^{\chi(v)}, G^{-\chi(v)} \notin \partial \mathcal{H}^{\chi(v)} \oplus \partial \mathcal{H}^{-\chi(v)}$ ). So we prefer a better formulation of the problem.

**Problem  $G_n$ .** Let  $G, g \in C^\alpha(\partial_0 \mathbb{D}^n, \mathbb{C})$ ,  $0 < \alpha < 1$ . For a fixed  $0 \leq v \leq n$  find functions  $\phi^{\chi(v)}, \phi^{-\chi(v)}$  analytic in  $\mathbb{D}^{\chi(v)}, \mathbb{D}^{-\chi(v)}$ , respectively, such that

$$\phi^{\chi(v)}(\zeta) + \phi^{-\chi(v)}(\zeta) G^{\chi(v)}(\zeta) = g^{\chi(v)}(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n, \quad (36)$$

where  $G^{\chi(v)}(\zeta) = G(\zeta)|_{\partial \mathcal{H}^{\chi(v)} \oplus \partial \mathcal{H}^{-\chi(v)}}$  with  $G^{\chi(v)}(\zeta) \neq 0$  and  $g^{\chi(v)}(\zeta) = g(\zeta)|_{\partial \mathcal{H}^{\chi(v)} \oplus \partial \mathcal{H}^{-\chi(v)}}$ ,  $\zeta \in \partial_0 \mathbb{D}^n$ ,  $0 \leq v \leq n$ .

Now every function in Eq. (36) belongs to the same space  $\partial \mathcal{H}^{\chi(v)} \oplus \partial \mathcal{H}^{-\chi(v)}$  just like in the one variable case. Thus for solving Eq. (36) we do not need any restrictions. We already restricted everything in Eq. (36) to the space  $\partial \mathcal{H}^{\chi(v)} \oplus \partial \mathcal{H}^{-\chi(v)}$ . Hence, this is a well-posed formulation of the Riemann problem for the truly higher dimensional torus.

So far we have obtained the Plemelj–Sokhotzki formula for higher dimensional torus related domains, a better way of notation of functions and a well-posed formulation of the Riemann problem for the truly  $n(2 \leq n)$ -dimensional torus.

It is known that the zero set of analytic functions in higher dimensional space can be connected and that makes the application of the index method questionable for higher dimensional space in general. However, variables of analytic functions in torus domains are independent. So we can apply index methods of one variable to every variable while fixing the others. Connected sets of zeros is not a problem for torus related domains.

### 3. Canonical Riemann functions and the homogeneous problem

**Theorem 4.** The homogeneous problem (36) with  $g^{\chi(v)} = 0$  is nontrivially solvable if and only if

$$\text{sign}[K(\chi(v))] = \chi(v) \quad (37)$$

holds for  $K(\chi(v)) = (-K_{\sigma_1}, \dots, -K_{\sigma_v}, K_{\sigma_{v+1}}, \dots, K_{\sigma_n})$  with  $K_{\sigma_\tau} \geq 0$  for  $1 \leq \tau \leq n$ , i.e., the sign of the domain is the same as the sign of the index  $K(\chi(v))$ , where

$$K_{\sigma_i} := \left| \frac{1}{(2\pi i)} \int_{\partial \mathbb{D}_{\sigma_i}} d \log(-G^{\chi(v)}(\zeta)) \right| \in \mathbb{N} \cup \{0\}.$$



For  $K(\chi(v))$ , which satisfies (37), the homogeneous problem (36) with  $g^{\chi(v)} = 0$  has  $|K(\chi(v))| + 1$  linearly independent solutions

$$\begin{cases} \phi^{\chi(v)}(z) = z_{\sigma_1}^{-k_{\sigma_1}} \cdots z_{\sigma_v}^{-k_{\sigma_v}} z_{\sigma_{v+1}}^{k_{\sigma_{v+1}}} \cdots z_{\sigma_n}^{k_{\sigma_n}} e^{\gamma^{\chi(v)}(z)}, & z \in \mathbb{D}^{\chi(v)}, \\ \phi^{-\chi(v)}(z) = z_{\sigma_1}^{K_{\sigma_1}-k_{\sigma_1}} \cdots z_{\sigma_v}^{K_{\sigma_v}-k_{\sigma_v}} z_{\sigma_{v+1}}^{k_{\sigma_{v+1}}-K_{\sigma_{v+1}}} \cdots z_{\sigma_n}^{k_{\sigma_n}-K_{\sigma_n}} e^{\gamma^{-\chi(v)}(z)}, & z \in \mathbb{D}^{-\chi(v)}, \end{cases} \quad (38)$$

where  $0 \leq k_{\sigma_\tau} \leq K_{\sigma_\tau}$ ,  $1 \leq \tau \leq n$ , and

$$\gamma^{\pm\chi(v)}(z) := \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \log\{\zeta^{-K(\chi(v))}(-G^{\chi(v)}(\zeta))\} C(\zeta, z) \frac{d\zeta}{\zeta}, \quad z \in \mathbb{D}^{\pm\chi(v)}.$$

The canonical function is

$$\begin{cases} X^{\chi(v)}(z) = e^{\gamma^{\chi(v)}(z)}, & z \in \mathbb{D}^{\chi(v)}, \\ X^{-\chi(v)}(z) = z^{-K(\chi(v))} e^{\gamma^{-\chi(v)}(z)}, & z \in \mathbb{D}^{-\chi(v)}. \end{cases} \quad (39)$$

The general solution to the homogeneous problem is

$$\begin{cases} \phi^{\chi(v)}(z) = P_{K(\chi(v))}(z) X^{\chi(v)}(z), & z \in \mathbb{D}^{\chi(v)}, \\ \phi^{-\chi(v)}(z) = P_{K(\chi(v))}(z) X^{-\chi(v)}(z), & z \in \mathbb{D}^{-\chi(v)}, \end{cases} \quad (40)$$

where  $P_{K(\chi(v))}(z)$  is a polynomial of  $z \in \mathbb{D}^{\chi(v)} = \mathbb{D}_{\sigma_1}^- \times \cdots \times \mathbb{D}_{\sigma_v}^- \times \mathbb{D}_{\sigma_{v+1}}^+ \times \cdots \times \mathbb{D}_{\sigma_n}^+$  with degree up to  $K(\chi(v))$  with arbitrary coefficients.

**Remark.** The solution (40) can be represented also as

$$\begin{cases} \phi^{\chi(v)}(z) = P_{K^+(\chi(v))}^{\chi(v)}(z) e^{\gamma^{\chi(v)}(z)}, & z \in \mathbb{D}^{\chi(v)}, \\ \phi^{-\chi(v)}(z) = P_{K^+(\chi(v))}^{-\chi(v)}(z) e^{\gamma^{-\chi(v)}(z)}, & z \in \mathbb{D}^{-\chi(v)}, \end{cases} \quad (41)$$

where  $K^+(\chi(v)) = (K_{\sigma_1}, \dots, K_{\sigma_v}, K_{\sigma_{v+1}}, \dots, K_{\sigma_n})$  and  $P_{K^+(\chi(v))}^{\chi(v)}(z)$  represents a polynomial with degree at most  $K^+(\chi(v))$  in  $\mathbb{D}^{\chi(v)}$ . The polynomials  $P_{K^+(\chi(v))}^{\pm\chi(v)}(z)$  in (41) are polynomials with arbitrary coefficients. Thus if we denote

$$\begin{cases} X_0^{\chi(v)}(z) = e^{\gamma^{\chi(v)}(z)}, & z \in \mathbb{D}^{\chi(v)}, \\ X_0^{-\chi(v)}(z) = e^{\gamma^{-\chi(v)}(z)}, & z \in \mathbb{D}^{-\chi(v)}, \end{cases} \quad (42)$$

as *simple canonical function of the Riemann problem*, then the solution to the homogeneous problem is

$$\begin{cases} \phi^{\chi(v)}(z) = P_{K^+(\chi(v))}^{\chi(v)}(z) X_0^{\chi(v)}(z), & z \in \mathbb{D}^{\chi(v)}, \\ \phi^{-\chi(v)}(z) = P_{K^+(\chi(v))}^{-\chi(v)}(z) X_0^{-\chi(v)}(z), & z \in \mathbb{D}^{-\chi(v)}. \end{cases} \quad (43)$$

**Proof.** It is known that the variables of the torus domains are independent. If the condition (37) does not hold for one variable  $z_\ell$  among  $(z_1, \dots, z_n)$ , then problem (36) due to Theorem 1 is not solvable for this variable.

Due to the fact that variables of analytic functions in torus domains are independent and the components of the multi-index are calculated independently, without worrying about possible connected zeros of analytic functions we can apply some part of the one-dimensional technique [4] to solve the problem.

Suppose that condition (37) holds. We look at two arbitrary components  $\zeta_{\sigma_\mu}$ ,  $1 \leq \mu \leq \nu$ , and  $\zeta_{\sigma_\tau}$ ,  $\nu + 1 \leq \tau \leq n$ , of  $\zeta \in \partial_0 \mathbb{D}^{\chi(v)}$ . Let  $N_{\sigma_\mu}^-$  be the number of zeros of  $\phi^{\chi(v)}(\zeta_{\sigma_\mu}, \cdot)$  and  $N_{\sigma_\tau}^+$  be the number of zeros of  $\phi^{\chi(v)}(\zeta_{\sigma_\tau}, \cdot)$ . Clearly

$$\begin{aligned} -N_{\sigma_\mu}^- &= \text{index } \phi^{\chi(v)}(\zeta_{\sigma_\mu}, \cdot) = \text{index}(-G(\zeta_{\sigma_\mu}, \cdot)\phi^{-\chi(v)}(\zeta_{\sigma_\mu}, \cdot)) \\ &= \text{index}(-G(\zeta_{\sigma_\mu}, \cdot)) + \text{index}(\phi^{-\chi(v)}(\zeta_{\sigma_\mu}, \cdot)) = -K_{\sigma_\mu} + N_{\sigma_\mu}^+, \quad 1 \leq \mu \leq \nu, \\ N_{\sigma_\tau}^+ &= \text{index } \phi^{\chi(v)}(\zeta_{\sigma_\tau}, \cdot) = \text{index}(-G(\zeta_{\sigma_\tau}, \cdot)\phi^{-\chi(v)}(\zeta_{\sigma_\tau}, \cdot)) \\ &= \text{index}(-G(\zeta_{\sigma_\tau}, \cdot)) + \text{index}(\phi^{-\chi(v)}(\zeta_{\sigma_\tau}, \cdot)) = K_{\sigma_\tau} - N_{\sigma_\tau}^-, \quad \nu + 1 \leq \tau \leq n, \end{aligned}$$

i.e.,  $K_{\sigma_\mu} = N_{\sigma_\mu}^+ + N_{\sigma_\mu}^- \geq 0$ ,  $K_{\sigma_\tau} = N_{\sigma_\tau}^+ + N_{\sigma_\tau}^- \geq 0$ .

If  $K(\chi(v)) = 0$  then  $N_{\sigma_\tau}^+ = N_{\sigma_\tau}^- = 0$  for all  $1 \leq \tau \leq n$  and  $\phi^{\pm\chi(v)}$  is a single-valued analytic function in every component  $z_{\sigma_\tau}$  of  $z \in \mathbb{D}^{\chi(v)}$ . Then the function  $\log(-G^{\chi(v)}(\zeta))$  is also single-valued [4].

Taking log from both sides of (36) with  $g^{\chi(v)} = 0$  we have

$$\log \phi^{\chi(v)}(\zeta) - \log \phi^{-\chi(v)}(\zeta) = \log(-G^{\chi(v)}(\zeta)), \quad \zeta \in \partial_0 \mathbb{D}^n. \quad (44)$$

Clearly  $\log(-G^{\chi(v)}(\zeta))$  satisfies the conditions of the Plemelj–Sokhotzki formula (30). Hence we can apply the formula (30) to this equation. However, between the formula and the equation depending on the values of  $n$  and  $\nu$  a difference about the sign of  $\log \phi^{-\chi(v)}(\zeta)$  could appear. If there is no difference we apply the Plemelj–Sokhotzki formula (30) to this equation directly. If there is a difference we replace  $\log \phi^{\chi(v)}(\zeta)$  and  $\log \phi^{-\chi(v)}(\zeta)$  with  $\log \phi_0^{\chi(v)}(\zeta)$  and  $-\log \phi_0^{-\chi(v)}(\zeta)$  and then apply the Plemelj–Sokhotzki formula (30) to the equation of  $\log \phi_0^{\chi(v)}(\zeta)$  and  $-\log \phi_0^{-\chi(v)}(\zeta)$ . Afterwards we can go back to the original solution. So having this fact in mind we neglect the possible difference of sign and represent the solutions of (44) by the Cauchy integral

$$\log \phi^{\pm\chi(v)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \log(-G^{\chi(v)}(\zeta)) C(\zeta, z) \frac{d\zeta}{\zeta} =: \gamma^{\pm\chi(v)}(z), \quad z \in \mathbb{D}^{\pm\chi(v)}. \quad (45)$$

Thus  $\phi^{\chi(v)}(z) = e^{\gamma^{\chi(v)}(z)}$ , i.e.,  $\phi^{\pm\chi(v)}(z) = e^{\gamma^{\pm\chi(v)}(z)}$ ,  $z \in \mathbb{D}^{\pm\chi(v)}$ , satisfies (36) with  $g^{\chi(v)} = 0$  on  $\partial_0 \mathbb{D}^n$ . Moreover, the values of  $\phi^{\pm\chi(v)}(z)$  at partial (componentwise) infinity can vary from a constant to some analytic functions of some components of  $z \in \mathbb{D}^{\pm\chi(v)}$  just as (29).

The case  $K_{\sigma_\tau} > 0$  for some or all  $1 \leq \tau \leq n$  can be proved similarly. For details one can look at e.g. [4].  $\square$

### 3.1. The inhomogeneous problem

**Theorem 5.** *If the sign of the index  $[K(\chi(v))]$  of  $G^{\chi(v)}(\zeta)$  in (36) is exactly the same as  $\chi(v)$ , the solution to the problem can be given by*

$$\phi^{\pm\chi(v)}(z) = X^{\pm\chi(v)}(z) [\psi^{\pm\chi(v)}(z) + P_{K^+(\chi(v))}^{\pm\chi(v)}(z)], \quad z \in \mathbb{D}^{\pm\chi(v)}. \quad (46)$$

If the sign $[K(\chi(v))]$  has  $\tau + \mu$  ( $0 \leq \tau \leq v$ ,  $0 \leq \mu \leq n - v$ ,  $0 < \mu + \tau \leq n$ ) opposite components compared with  $\chi(v)$  (i.e.,  $K_{\sigma_i} < 0$  ( $1 \leq i \leq \tau \leq v$ ),  $K_{\sigma_{v+j}} < 0$  ( $1 \leq j \leq \mu \leq n - v$ ) and the remaining  $K'_{\sigma_i}$  are nonnegative), the solvability condition

$$\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{g^{\chi(v)}(\zeta)}{X^{\chi(v)}(\zeta)} \prod_{\alpha=1}^{\tau} \bar{\zeta}_{\sigma_{\alpha}}^{k_{\sigma_{\alpha}}} \prod_{\beta=\tau+1}^v \bar{\zeta}_{\sigma_{\beta}}^{k_{\sigma_{\beta}}} \prod_{j=v+1}^{v+\mu} \zeta_{\sigma_j}^{k_{\sigma_j}} \prod_{\theta=v+\mu+1}^n \zeta_{\sigma_{\theta}}^{k_{\sigma_{\theta}}} \frac{d\zeta}{\zeta} = 0 \quad (47)$$

with

$$0 \leq k_{\sigma_{\alpha}} \leq -K_{\sigma_{\alpha}} \quad (1 \leq \alpha \leq \tau), \quad 0 \leq k_{\sigma_j} \leq -K_{\sigma_j} \quad (v+1 \leq j \leq v+\mu),$$

$$0 \leq \sum_{\alpha=1}^{\tau} k_{\sigma_{\alpha}} + \sum_{j=v+1}^{v+\mu} k_{\sigma_j} \leq -\sum_{\alpha=1}^{\tau} K_{\sigma_{\alpha}} - \sum_{j=v+1}^{v+\mu} K_{\sigma_j} - 1, \quad k_{\sigma_{\beta}}, k_{\sigma_{\theta}} \in \mathbb{Z}_+$$

must be satisfied. Then the solution is

$$\phi^{\pm\chi(v)}(z) = X^{\pm\chi(v)}(z) \psi^{\pm\chi(v)}(z), \quad z \in \mathbb{D}^{\pm\chi(v)}, \quad (48)$$

where

$$\psi^{\pm\chi(v)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} [g^{\chi(v)}(\zeta)/X^{\chi(v)}(\zeta)] C(\zeta, z) \frac{d\zeta}{\zeta}, \quad z \in \mathbb{D}^{\pm\chi(v)}. \quad (49)$$

**Proof.** Suppose condition (37) is satisfied. Dividing (36) by  $X^{\chi(v)}$  gives

$$\frac{\phi^{\chi(v)}(\zeta)}{X^{\chi(v)}(\zeta)} + \frac{\phi^{-\chi(v)}(\zeta)}{X^{-\chi(v)}(\zeta)} = \frac{g^{\chi(v)}(\zeta)}{X^{\chi(v)}(\zeta)}, \quad \zeta \in \partial_0 \mathbb{D}^n, \quad (50)$$

where  $X^{\chi(v)}(\zeta) = G^{\chi(v)}(\zeta) X^{-\chi(v)}(\zeta)$ ,  $\zeta \in \partial_0 \mathbb{D}^n$  is applied.

Because  $g^{\chi(v)} \in C^{\alpha}(\partial_0 \mathbb{D}^n, \mathbb{C})$ ,  $0 < \alpha < 1$ , it is easy to see that  $X^{\chi(v)} \in C^{\alpha}(\partial_0 \mathbb{D}^n, \mathbb{C})$ ,  $0 < \alpha < 1$ , and therefore  $[g^{\chi(v)}(\zeta)/X^{\chi(v)}(\zeta)] \in C^{\alpha}(\partial_0 \mathbb{D}^n, \mathbb{C})$ ,  $0 < \alpha < 1$ , see [4]. Again due to  $G^{\chi(v)}, g^{\chi(v)}, X^{\pm\chi(v)} \in \partial\mathcal{H}^{\chi(v)} \oplus \partial\mathcal{H}^{-\chi(v)}$  it is not difficult to prove that  $[g^{\chi(v)}(\zeta)/X^{\chi(v)}(\zeta)] \in \partial\mathcal{H}^{\chi(v)} \oplus \partial\mathcal{H}^{-\chi(v)}$ .

Thus Eq. (50) satisfies the conditions of the Plemelj–Sokhotski formula (30) if we neglect the fact that there could be a trivial argument about the difference of one sign as we have mentioned in the homogeneous case. So the solution to this problem can be given by (49). Therefore  $\psi^{\pm\chi(v)}$  satisfies

$$\psi^{\chi(v)}(\zeta) + \psi^{-\chi(v)}(\zeta) = \frac{g^{\chi(v)}(\zeta)}{X^{\chi(v)}(\zeta)}, \quad \zeta \in \partial_0 \mathbb{D}^n.$$

Hence because  $[\phi^{\chi(v)}/X^{\chi(v)}] - \psi^{\pm\chi(v)}$  is continuous on  $\partial_0 \mathbb{D}^n$  we have

$$\frac{\phi^{\chi(v)}}{X^{\chi(v)}} - \psi^{\chi(v)} = \frac{\phi^{-\chi(v)}}{X^{-\chi(v)}} - \psi^{-\chi(v)} \quad \text{on } \partial_0 \mathbb{D}^n.$$

Again  $[\phi^{\chi(v)}/X^{\chi(v)}] - \psi^{\chi(v)}$  is an analytic function in  $\mathbb{D}_{\sigma_1}^- \times \cdots \times \mathbb{D}_{\sigma_v}^- \times \mathbb{D}_{\sigma_{v+1}}^+ \times \cdots \times \mathbb{D}_{\sigma_n}^+$  and  $[\phi^{-\chi(v)}/X^{-\chi(v)}] - \psi^{-\chi(v)}$  is analytic in  $\mathbb{D}_{\sigma_1}^+ \times \cdots \times \mathbb{D}_{\sigma_v}^+ \times \mathbb{D}_{\sigma_{v+1}}^- \times \cdots \times \mathbb{D}_{\sigma_n}^-$  having a pole at  $(z_{\sigma_1} = 0, \dots, z_{\sigma_v} = 0, z_{\sigma_{v+1}} = \infty, \dots, z_{\sigma_n} = \infty)$  of order  $K(\chi(v))$ , i.e.,  $[\phi^{-\chi(v)}/X^{-\chi(v)}] - \psi^{-\chi(v)}$  could have at most a term like  $z_{\sigma_1}^{-K_{\sigma_1}} \cdots z_{\sigma_v}^{-K_{\sigma_v}} z_{\sigma_{v+1}}^{K_{\sigma_{v+1}}} \cdots z_{\sigma_n}^{K_{\sigma_n}}$  for  $|z_{\sigma_1}| < 1, \dots, |z_{\sigma_v}| < 1, |z_{\sigma_{v+1}}| > 1, \dots, |z_{\sigma_n}| > 1$ .

If the  $\text{sign}[K(\chi(v))]$  has  $\tau + \mu$  ( $1 \leq \tau + \mu \leq n$ ) opposite components compared with  $\chi(v)$ , we can expect

$$\begin{cases} X^{\chi(v)}(z) = e^{\gamma^{\chi(v)}(z)}, & z \in \mathbb{D}^{\chi(v)}, \\ X^{-\chi(v)}(z) = \prod_{i=1}^{\tau} z_{\sigma_i}^{K_{\sigma_i}} \prod_{i=\tau+1}^v z_{\sigma_i}^{K_{\sigma_i}} \prod_{i=v+1}^{v+\mu} z_{\sigma_i}^{-K_{\sigma_i}} \prod_{i=v+\mu+1}^n z_{\sigma_i}^{-K_{\sigma_i}} e^{\gamma^{-\chi(v)}(z)}, \\ z \in \mathbb{D}^{-\chi(v)}, \end{cases} \quad (51)$$

where  $K_{\sigma_i} < 0$  ( $1 \leq i \leq \tau \leq v$ ),  $K_{\sigma_{v+j}} < 0$  ( $1 \leq j \leq \mu \leq n - v$ ) and the remaining  $K'_{\sigma_i}$ s are nonnegative.

In this case due to the factor  $z_{\sigma_1}^{K_{\sigma_1}} \cdots z_{\sigma_{\tau}}^{K_{\sigma_{\tau}}} z_{\sigma_{v+1}}^{-K_{\sigma_{v+1}}} \cdots z_{\sigma_{v+\mu}}^{-K_{\sigma_{v+\mu}}}$  of  $X^{-\chi(v)}(z)$  the function  $\psi^{-\chi(v)}(z)$  cannot be an arbitrary holomorphic function in  $\mathbb{D}^{-\chi(v)}$  if  $\phi^{-\chi(v)}(z) = X^{-\chi(v)}(z)\psi^{-\chi(v)}(z)$  has to be holomorphic in  $\mathbb{D}^{-\chi(v)}$ . Therefore it is necessary and sufficient that  $\psi^{-\chi(v)}(z)$  does not include all the terms  $z_{\sigma_i}$  ( $1 \leq i \leq \tau$ ) up to  $-K_{\sigma_i}$  and the terms  $z_{\sigma_{v+j}}$  ( $1 \leq j \leq \mu$ ) up to  $-K_{\sigma_{v+j}}$ . These conditions may be expressed in terms of (47) by looking at the coefficients of the Fourier series of the function  $\psi^{-\chi(v)}(z)$  or equally in the form of

$$\begin{aligned} \psi^{-\chi(v)}(\zeta) &= \zeta_{\sigma_1}^{-K_{\sigma_1}} \cdots \zeta_{\sigma_{\tau}}^{-K_{\sigma_{\tau}}} \zeta_{\sigma_{v+1}}^{K_{\sigma_{v+1}}} \cdots \zeta_{\sigma_{v+\mu}}^{K_{\sigma_{v+\mu}}} \varphi(\zeta), \\ \varphi(\zeta) &\in \partial \mathcal{H}^{\chi_{\sigma_1} \cdots \chi_{\sigma_v}}(v) \oplus \partial \mathcal{H}^{-\chi_{\sigma_1} \cdots \chi_{\sigma_v}}(v). \end{aligned} \quad (52)$$

Thus if  $\tau = \mu = 0$  then  $\text{sign}[K^+(\chi(v))] = \chi(v)$ , i.e., they have the same sign components and the solution is given by (46) without condition. If  $\tau + \mu > 0$  then the solution is given by (48) with the solvability condition (47).  $\square$

If  $\tau = v$  and  $\mu = n - v$ , i.e., if  $\tau + \mu = n$ , then  $\text{sign}[K(\chi(v))] = -\chi(v)$ , i.e., they both have totally opposite signs. In this case we have the following.

**Lemma 2.** *If  $\text{sign}[K(\chi(v))] = -\chi(v)$ , then the solvability condition*

$$\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{g^{\chi(v)}(\zeta)}{X^{\chi(v)}(\zeta)} \prod_{\alpha=1}^v \bar{\zeta}_{\sigma_{\alpha}}^{-k_{\sigma_{\alpha}}} \prod_{j=v+1}^n \zeta_{\sigma_j}^{k_{\sigma_j}} \frac{d\zeta}{\zeta} = 0 \quad (53)$$

with

$$\begin{aligned} 0 \leq k_{\sigma_{\alpha}} \leq -K_{\sigma_{\alpha}} \quad (1 \leq \alpha \leq v), \quad 0 \leq k_{\sigma_j} \leq -K_{\sigma_j} \quad (v+1 \leq j \leq n), \\ 0 \leq \sum_{\alpha=1}^v k_{\sigma_{\alpha}} + \sum_{j=v+1}^n k_{\sigma_j} \leq -\sum_{\alpha=1}^v K_{\sigma_{\alpha}} - \sum_{j=v+1}^n K_{\sigma_j} - 1 \end{aligned}$$

must be satisfied. Then the solution to problem (36) is (48).

Specially, if  $v = 0$  and  $\text{sign}[K(\chi(0))] = -\chi(0)$ , i.e.,  $K(\chi(0)) = (-K_1, \dots, -K_n)$  with  $K_i \geq 0$ , then a finite number of solvability conditions, namely

$$\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{g^{\chi(0)}(\zeta)}{X^{\chi(0)}(\zeta)} \zeta^{\mathbf{k}} \frac{d\zeta}{\zeta} = 0, \quad \mathbf{0} \leq \mathbf{k} \leq -\mathbf{K}(\chi(0)) - \mathbf{e}_i, \quad i = 1, \dots, n, \quad (54)$$

must be satisfied, where  $\mathbf{0} = (0, \dots, 0)$ ,  $\mathbf{k} = (k_1, \dots, k_n)$ ,

$$\mathbf{e}_i = (\underbrace{0, \dots, 0}_i, 1, 0, \dots, 0), \quad 0 \leq \sum_{j=1}^n k_j \leq -\sum_{j=1}^n K_j - 1.$$

Then the solution to problem (36) is (48).

### 3.2. The connection between the Riemann problem and the Riemann–Hilbert problem

To our knowledge no study has been made yet about the connection between the Riemann problem and the Riemann–Hilbert problem for polydomains. A unique relevant study in this aspect on the Riemann–Hilbert problem in the polydisc  $\mathbb{D}^n$  is made by [3]. However this study is just for one pair of all torus domains, for the rest the Riemann–Hilbert problem is still open.

Through this connection problem one can see that without making some restrictions on the form of analytic functions, the conversion between these two problems simply is impossible on the basis of the former studies. This obstacle could be overcome by a slight modification of the corresponding Cauchy kernel [1,20] and thus the conversion works well due to the fact that the reflection principle holds for boundary values of analytic functions in every pair of torus domains by the new classification.

The Riemann–Hilbert problem for analytic functions and for inhomogeneous Cauchy–Riemann systems in polydiscs were considered in [3]. One has to find  $\phi$  from the general Riemann–Hilbert boundary value problem

$$\operatorname{Re}\{\overline{\lambda(\zeta)}\phi(\zeta)\} = \varphi(\zeta), \quad \zeta \in \partial_0\mathbb{D}^n,$$

for  $\lambda \in C(\partial_0\mathbb{D}^n, \mathbb{C})$ ,  $|\lambda(\zeta)| = 1$ ,  $\zeta \in \partial_0\mathbb{D}^n$ .

In order this equation to be solvable  $\lambda$  and  $\varphi$  have to satisfy some solvability conditions. Calculation of these solvability conditions needs much effort. Clearly this is not a well-posed formulation. Thus we at first give a well-posed formulation of the Riemann–Hilbert boundary value problem for torus domains and then mention some facts about the connection to the Riemann problem.

**Problem  $G_n(\mathbf{C})$ .** Let  $\lambda \in C^\alpha(\partial_0\mathbb{D}^n, \mathbb{C})$ ,  $\varphi \in C^\alpha(\partial_0\mathbb{D}^n, \mathbb{R})$ ,  $0 < \alpha < 1$ . Find an analytic function  $\phi^{\chi(v)}$  ( $0 \leq v \leq n$ ) in  $\mathbb{D}^{\chi(v)}$  such that

$$\operatorname{Re}\{\overline{\lambda^{\chi(v)}(\zeta)}\phi^{\chi(v)}(\zeta)\} = \varphi^{\chi(v)}(\zeta), \quad \zeta \in \partial_0\mathbb{D}^n, \quad (55)$$

where  $\lambda^{\chi(v)}(\zeta) := \lambda(\zeta)|_{\partial\mathcal{H}^{\chi(v)} \oplus \partial\mathcal{H}^{-\chi(v)}}$  with  $|\lambda^{\chi(v)}(\zeta)| = 1$  and  $\varphi^{\chi(v)}(\zeta) := \varphi(\zeta)|_{\partial\mathcal{H}^{\chi(v)} \oplus \partial\mathcal{H}^{-\chi(v)}}$  for  $\zeta \in \partial_0\mathbb{D}^n$ .

The reason why this is a well-posed formulation and why we formulate the problem in this way is the same as (36). So we omit the explanation.

We have mentioned that for analytic functions defined by the modified Cauchy kernel (28) the relationship

$$\overline{(-1)^v \phi^{\chi(v)}(\zeta)} = (-1)^v \phi^{-\chi(v)}(\zeta), \quad \zeta \in \partial_0\mathbb{D}^n, \quad 0 \leq v \leq n,$$

holds for  $\varphi(\eta)$  real on  $\partial_0\mathbb{D}^n$  and  $\varphi(0) = 0$  (without  $\varphi(0) = 0$  we have one free parameter to fix). Therefore with the transformation from (36) to (55) we do not need to put any restriction on the form of analytic functions, i.e., we have not to cut out some branches of analytic functions to get the transformation as all known studies have to do if they try to establish the connection. This

shows again the impact of the modified Cauchy kernel (28) and that our division of the given boundary values is properly made.

**Theorem 6.** *The solution of the Riemann problem (36) with*

$$G^{\chi(v)}(\zeta) = \frac{\lambda^{\chi(v)}(\zeta)}{\overline{\lambda^{\chi(v)}(\zeta)}}, \quad g^{\chi(v)}(\zeta) = \frac{2\varphi^{\chi(v)}(\zeta)}{\overline{\lambda^{\chi(v)}(\zeta)}}$$

*is a solution of the Riemann–Hilbert problem (55), if some free complex parameters are chosen properly.*

We have the solutions of the Riemann problem (36). However for the Riemann–Hilbert problem (55) we have the solution only for  $\nu = 0$  see [3], for the other  $\nu$ ,  $1 \leq \nu \leq n$ , the solution to the Riemann–Hilbert problem (55) has to be found. It will be given in another paper. Thus the proof is omitted here.

## References

- [1] H. Begehr, A. Mohammed, The Schwarz problem for analytic functions in torus related domains, *Appl. Anal.*, in press.
- [2] H. Begehr, D.Q. Dai, Spatial Riemann problem for analytic functions of two complex variables, *J. Anal. Appl.* 18 (1999) 827–837.
- [3] H. Begehr, A. Dzhruev, *An Introduction to Several Complex Variables and Partial Differential Equations*, Pitman Monogr. Surv. Pure Appl. Math., vol. 88, Addison–Wesley/Longman, Harlow, 1997.
- [4] H. Begehr, *Complex Analytic Methods for Partial Differential Equations*, World Scientific, London, 1994.
- [5] H. Begehr, G.C. Wen, *Nonlinear Elliptic Boundary Value Problems and Their Applications*, Longman, Harlow, 1996.
- [6] J.W. Cohen, O.J. Boxma, *Boundary Value Problems in Queuing System Analysis*, North-Holland Math. Stud., vol. 79, Elsevier, Amsterdam, 1983.
- [7] Persy Deift, *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach*, Courant Lect. Notes Math., vol. 3, Courant Inst. Math. Sci., New York Univ., New York, 2000.
- [8] Dao Qing Dai, Fourier method for an over-determined elliptic system with several complex variables, *Acta Math. Sin. (Engl. Ser.)* 22 (1) (2006) 87–94.
- [9] F.D. Gakhov, *Boundary Value Problems*, Fizmatgiz, Moscow, 1963 (in Russian); English translation: Pergamon Press, Oxford, 1966.
- [10] V.A. Kakichev, Boundary value problems of linear conjugation for functions holomorphic in bicylindrical regions, *Soviet Math. Dokl.* 9 (1968) 222–226.
- [11] V.A. Kakichev, Analysis of conditions of solvability for a class of spatial Riemann problems, *Ukrainian J. Math.* 31 (1979) 205–210 (English translation).
- [12] V.A. Kakichev, Application of the Fourier method to the solution of boundary value problems for functions analytic in disk bidomains, *Amer. Math. Soc. Transl. Ser. 2* 146 (1990) 33–41.
- [13] S.G. Krantz, *Complex Analysis: The Geometric Viewpoint*, Carus Math. Monogr., Math. Assoc. America, 1990.
- [14] A. Kufner, J. Kadlec, *Fourier Series*, Iliffe Books, Academia, Prague, 1971.
- [15] A. Kumar, A generalized Riemann boundary problem in two variables, *Arch. Math.* 62 (1994) 531–538.
- [16] Xing Li, An application of the periodic Riemann boundary value problem to a periodic crack problem, in: H. Begehr, A. Celebi, W. Tutschke (Eds.), *Complex Methods for Partial Differential and Integral Equations*, Kluwer Academic Publ., Dordrecht, 1999, pp. 103–112.
- [17] Chien-ke Lu, *Boundary Value Problems for Analytic Functions*, World Scientific, Singapore, 1993.
- [18] Chien-ke Lu, Periodic Riemann boundary value problems and their applications in elasticity, *Chinese Mathematics* 4 (1964) 372–422.
- [19] V.G. Maz'ya, S.M. Nikol'skii, *Analysis IV*, Encyclopedia Math. Sci., vol. 27, Springer-Verlag, Berlin, 1991.
- [20] Alip Mohammed, Boundary value problems of complex variables, PhD thesis, FU Berlin 2002, <http://www.diss.fu-berlin.de/2003/23/indexe.html>.

- [21] Alip Mohammed, The Neumann problem for the inhomogeneous pluriharmonic system in polydiscs, in: H. Begehr, A. Celebi, W. Tutschke (Eds.), *Complex Methods for Partial Differential and Integral Equations*, Kluwer Academic Publ., Dordrecht, 1999, pp. 155–164.
- [22] N.I. Muskhelishvili, *Singular Integral Equations*, Fizmatgiz, Moscow, 1946 (in Russian); English translation: Noordhoff, Groningen, 1953.
- [23] V.S. Vladimirov, Problems of linear conjugacy of holomorphic functions of several complex variables, *Trans. Amer. Math. Soc.* 72 (1969) 203–232.